Queen Mary University of London

B. Sc. Examination by course unit 2015

MTH6140: Linear Algebra II

Duration: 2 hours

Date and time: 7th May 2015, 14:30–16:30

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You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Examiner(s): T. W. Müller

Question 1 (20 marks). (a) State and prove the *Steinitz Exchange Lemma*. You may use (without proof) the fact that a homogeneous system of linear equations over a field with more variables than equations has a non-zero solution.

[10]

(b) Let V be a vector space with a finite spanning set over the field K.

(i) Show that V has a basis. **[3]**

(ii) Prove that any two bases of V have the same number of elements. **[4]**

(iii) Define the dimension of V. Explain carefully why this definition makes sense. **[3]**

Question 2 (20 marks). Let K and L be two fields such that $K \subseteq L$ (think for instance of the pair $\mathbb{Q} \subseteq \mathbb{R}$). In this situation, L may be viewed as a vector space over the field K, with vector addition given by addition in L and scalar multiplication being defined via the multiplication in L. Denote by

$$[L : K] = \dim_K L$$

the dimension of L as a vector space over K.

(a) Let K, L, M be three fields such that $K \subseteq L \subseteq M$, and suppose that $[L : K]$ and $[M : L]$ are both finite. Show that in this situation

$$[M : K] = [L : K] \cdot [M : L].$$

Hint: Write down a basis for the space L over the field K, and one for the space M over the field L, Then form all $[L : K] \cdot [M : L]$ products in the field M with the first factor coming from the first basis, and the second factor taken from the second basis. Show that these products form a basis for M over K. **[10]**

(b) Let K, L be two fields such that $K \subseteq L$, and consider L as a vector space over K, as defined above. Show that multiplication by a fixed element $\lambda \in L$ defines a linear map $T_\lambda : L \to L$, $x \mapsto \lambda x$. For which λ is T_λ invertible?

[8]

(c) Compute $[\mathbb{C} : \mathbb{R}]$ by exhibiting a basis for $\mathbb{C}$ over $\mathbb{R}$.

[2]

Question 3 (20 marks). Let V be a vector space of finite dimension n over the field K, let $m \geq 1$ be an integer, and let $\{e_1, e_2, \dots, e_n\}$ be a basis of V.

(a) Define the concept of a *linear* m*-form on* V.

[3]

(b) When is a linear m-form on V called *alternating*?

[2]

(c) Show that an alternating m-form F_a on V changes sign when two of its arguments are interchanged. Deduce that

$$F_a(e_{\pi(j_1)}, \dots, e_{\pi(j_m)}) = \operatorname{sgn}(\pi) F_a(e_{j_1}, \dots, e_{j_m}), \tag{1}$$

where $\pi \in \operatorname{Sym}(\{j_1, \dots, j_m\})$ and $(j_1, \dots, j_m) \in \{1, \dots, n\}^m$.

[5]

(d) (i) Define the determinant $|A|$ of an $n \times n$ matrix $A = (a_{i,j})$ over the field K. Which property of a determinant follows from the assertion proved in Part (c)?

[2]

(ii) Straight from the definition, evaluate the determinant $D = |A|$, where

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}.$$

[3]

(iii) State the *Laplace expansion theorem* for determinants, and use it to recompute the determinant D of Part (ii).

[5]

Question 4 (20 marks). (a) Derive Cramér's rule for the solution of an invertible $(n \times n)$-system of linear equations from results of the course.

[8]

(b) (i) When is a linear map $P : V \to V$ on a vector space V called a *projection*?

[3]

(ii) Suppose $U_1, \ldots, U_r \leq V$. What does it mean to say that the vector space V is the direct sum of $U_1, \ldots, U_r$?

[3]

(c) Show the following: if $P : V \to V$ is a projection, then we have

$$V = \mathrm{image}(P) \oplus \mathrm{kernel}(P).$$

[6]

Question 5 (20 marks). Let $\mathcal{C} = \mathcal{C}[0,\pi]$ be the set of continuous functions $f : [0,\pi] \to \mathbb{R}$. In what follows, you may quote standard theorems from calculus without proof. You may also use without proof the facts that

$$\int \sin(x)\,dx = -\cos(x),$$
$$\int \cos(x)\,dx = \sin(x),$$
$$\int \sin(x)\cos(x)\,dx = \frac{1}{2}\sin^2(x),$$
$$\int \sin^2(x)\,dx = \frac{1}{2}\big(x - \sin(x)\cos(x)\big),$$
$$\int \cos^2(x)\,dx = \frac{1}{2}\big(x + \sin(x)\cos(x)\big).$$

(a) Defining addition and scalar multiplication of functions via

$$(f+g)(x) = f(x) + g(x), \quad (f, g \in \mathcal{C},\ 0 \leq x \leq \pi),$$
$$(\alpha f)(x) = \alpha f(x), \quad (f \in \mathcal{C},\ \alpha \in,\ 0 \leq x \leq \pi),$$

show that $\mathcal{C}$ becomes a real vector space.

[3]

(b) What is an *inner product* on a real vector space V?

[2]

(c) Show that the function $\langle\cdot|\cdot\rangle : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ given by

$$\langle f|g\rangle := \int_0^\pi f(x)g(x)dx$$

is an inner product on the real vector space $\mathcal{C}$ as defined in Part (a).

[2]

(d) Apply the Gram-Schmidt algorithm to obtain an orthonormal system $\{g_1(x), g_2(x), g_3(x)\}$, such that

$$\big\langle g_1(x),\ g_2(x),\ g_3(x)\big\rangle = \big\langle \sin(x),\ \cos(x),\ 1\big\rangle.$$

[13]

End of Paper.