Queen Mary
University of London

B. Sc. Examination by course unit 2015

MTH6136: Statistical Theory

Duration: 2 hours

Date and time: 18th May 2015, 14:30–16:30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Electronic calculators may be used. The make and model should be specified on the script. The calculator must not be programmed (other than by the manufacturer) prior to the examination.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Examiner(s): B. M. Parker

Question 1. Suppose that $Y_1, \ldots, Y_n$ are independent Pareto random variables with probability density function

$$f_Y(y) = \frac{\theta 2^\theta}{y^{\theta+1}}, \qquad y \geqslant 2,$$

where $\theta > 0$.

(a) State Neyman's factorisation theorem. **[3]**

(b) Hence or otherwise, show that $\prod_{i=1}^n (Y_i)$ is a sufficient statistic for θ. **[5]**

(c) Show that the maximum likelihood estimator of θ is

$$\hat{\theta} = \frac{n}{\sum_{i=1}^n \log(Y_i) - n \log 2}.$$

[6]

(d) Evaluate the Cramér-Rao lower bound for unbiased estimators of θ. **[6]**

(e) Hence or otherwise, find an approximate 95% confidence interval for θ. **[4]**

Question 2. Let $Y_1, \ldots, Y_n$ be independent normal random variables with probability density function

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\} \qquad -\infty < y < \infty$$

where $-\infty < \mu < \infty$ and $\sigma^2 > 0$, and μ and σ^2 are unknown.

(a) Show that this distribution is a member of the exponential family. **[8]**

(b) Write down complete sufficient statistics for μ and σ^2. **[3]**

(c) Prove that $\bar{Y}$ is the minimum variance unbiased estimator of μ. **[5]**

Question 3. Let $Y_1, \ldots, Y_n$ be independent uniform random variables with probability density function

$$f_Y(y) = \frac{1}{\theta}, \quad 0 \leqslant y \leqslant \theta.$$

(a) Show that the method of moments estimator, $\tilde{\theta}$ of θ, is given by $\tilde{\theta} = 2\bar{Y}$. **[4]**

(b) Show that the mean and variance of the method of moments estimator are

$$\theta \quad \text{and} \quad \frac{\theta^2}{3n}$$

respectively. **[4]**

(c) Find the maximum likelihood estimator, $\hat{\theta}$ of θ. **[4]**

(d) Show that the pdf of the maximum likelihood estimator is

$$f_{\bar{\theta}}(y) = \frac{ny^{n-1}}{\theta^n}, \quad 0 < y < \theta\,.$$

[4]

(e) Show that the bias and variance of the maximum likelihood estimator are

$$\frac{-\theta}{n+1} \quad \text{and} \quad \frac{n\theta^2}{(n+1)^2(n+2)}$$

respectively. **[7]**

(f) Compute the mean square error of the two estimators $\tilde{\theta}$ and $\hat{\theta}$. For what values of n is the mean square error of $\hat{\theta}$ bigger than that of $\tilde{\theta}$? **[5]**

Question 4. Let $Y_1, \ldots, Y_{n_1}$ be exponential random variables with parameter λ_1 and let $Y_{n_1+1}, \ldots, Y_{n_1+n_2}$ be exponential random variables with parameter λ_2, all independent, where $\lambda_1 > 0$ and $\lambda_2 > 0$. Let $\bar{Y}_1$ be the mean of the first n_1 observations, and let $\bar{Y}_2$ be the mean of the remaining observations.

(a) Show that the maximum likelihood estimators of λ_1 and λ_2 are $\hat{\lambda}_1 = 1/\bar{Y}_1$ and $\hat{\lambda}_2 = 1/\bar{Y}_2$ and hence give the maximum likelihood estimator of λ_1/λ_2. **[6]**

(b) Given that $2\lambda_1 n_1 \bar{Y}_1$ and $2\lambda_2 n_2 \bar{Y}_2$ have chi-squared distributions with respective degrees of freedom $2n_1$ and $2n_2$, explain why $\lambda_1\bar{Y}_1/(\lambda_2\bar{Y}_2)$ is a pivot for λ_1/λ_2. **[6]**

(c) Use this pivot to derive an exact $100(1-\alpha)\%$ confidence interval for λ_1/λ_2. **[5]**

Question 5. Let $Y_1, \ldots, Y_n$ be independent mean zero normal random variables with probability density function

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{y^2}{2\sigma^2}\right\} \qquad -\infty < y < \infty$$

where $\sigma^2 > 0$, and consider testing $H_0 : \sigma = \sigma_0$ against $H_1 : \sigma = \sigma_1$ where $\sigma_1 > \sigma_0$ for fixed σ_0 and σ_1.

(a) Write down the likelihood, $L(\sigma^2; \underline{y})$, and hence find the generalised likelihood ratio given by $\Lambda(\underline{y}) = L(\sigma_0^2; \underline{y})/L(\sigma_1^2; \underline{y})$. **[6]**

(b) Find the general form of the most powerful test of H_0 against H_1. **[3]**

(c) Given that under H_0, $\sum_{i=1}^n Y_i^2/\sigma_0^2 \sim \chi_n^2$, derive the form of the critical region of the test with significance level α. **[2]**

(d) Explain why a uniformly most powerful test of $H_0 : \sigma = \sigma_0$ against $H_1 : \sigma = \sigma_1$ exists in this case. Describe briefly what is meant by a uniformly most powerful test. **[4]**

End of Paper.