Queen Mary
University of London

B. Sc. Examination by course unit 2014

MTH 6136 Statistical Theory

Duration: 2 hours

Date and time: 12 May 2014, 2.30-4.30

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Examiner(s): D. S. Coad and H. Grossmann

Question 1 Suppose that $Y_1, \ldots, Y_n$ are independent beta random variables with probability density function

$$f_Y(y) = \theta(1-y)^{\theta-1}, \quad 0 < y < 1,$$

where $\theta > 0$.

(a) Show that $\prod_{i=1}^n (1-Y_i)$ is a sufficient statistic for θ. [6]

(b) Evaluate the Cramér-Rao lower bound for unbiased estimators of $1/\theta$. [9]

(c) Given that $E\{\log(1-Y)\} = -1/\theta$ and that the above statistic is also complete, explain why $-\sum_{i=1}^n \log(1-Y_i)/n$ is the minimum variance unbiased estimator of $1/\theta$. [5]

Question 2 Let $Y_1, \ldots, Y_n$ be independent random variables from the gamma distribution with probability density function

$$f_Y(y) = \frac{\lambda^\alpha y^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda y}, \quad y > 0,$$

where $\alpha > 0$, $\lambda > 0$ and Γ denotes the gamma function.

(a) Express this distribution as a member of the exponential family. [7]

(b) State complete sufficient statistics for α and λ. [3]

(c) Given that $E(Y) = \alpha/\lambda$ and $\mathrm{var}(Y) = \alpha/\lambda^2$, derive the method of moments estimators of α and λ. Are these estimators efficient? [10]

Question 3 Suppose that $Y_1, \ldots, Y_n$ are independent Weibull random variables with second moment θ and probability density function

$$f_Y(y) = \frac{2y}{\theta} \exp\left(-\frac{y^2}{\theta}\right), \quad y > 0,$$

where $\theta > 0$.

(a) Show that the maximum likelihood estimator of θ is $\hat{\theta} = \sum_{i=1}^n Y_i^2/n$. [7]

(b) Find the asymptotic distribution of $\hat{\theta}$, and hence write down an approximate $100(1-\alpha)\%$ confidence interval for θ. [8]

(c) Given that the probability density function of $X = Y^2/\theta$ is $f_X(x) = \exp(-x)$ for $x > 0$, explain why $\sum_{i=1}^n Y_i^2/\theta$ is a pivot for θ and use this to obtain an exact $100(1-\alpha)\%$ confidence interval for θ. [5]

Question 4 Let $Y_1, \ldots, Y_{n_1}$ be normal random variables with zero mean and variance σ_1^2 and let $Y_{n_1+1}, \ldots, Y_{n_1+n_2}$ be normal random variables with zero mean and variance σ_2^2, all independent, where $\sigma_1^2 > 0$ and $\sigma_2^2 > 0$.

(a) Show that the maximum likelihood estimators of σ_1^2 and σ_2^2 are

$$\hat{\sigma}_1^2 = \frac{1}{n_1}\sum_{i=1}^{n_1} Y_i^2 \quad \text{and} \quad \hat{\sigma}_2^2 = \frac{1}{n_2}\sum_{i=n_1+1}^{n_1+n_2} Y_i^2,$$

and hence give the maximum likelihood estimator of σ_1^2/σ_2^2. [8]

(b) Given that $n_1\hat{\sigma}_1^2/\sigma_1^2$ and $n_2\hat{\sigma}_2^2/\sigma_2^2$ have chi-squared distributions with respective degrees of freedom n_1 and n_2, explain why $\sigma_2^2\hat{\sigma}_1^2/(\sigma_1^2\hat{\sigma}_2^2)$ is a pivot for σ_1^2/σ_2^2. [6]

(c) Use this pivot to derive an exact $100(1-\alpha)\%$ confidence interval for σ_1^2/σ_2^2. [6]

Question 5 Suppose that $Y_1, \ldots, Y_n$ are independent Pascal random variables with probability mass function

$$P(Y = y) = \begin{pmatrix} y + r - 1 \\ r - 1 \end{pmatrix} \pi^r (1-\pi)^y, \qquad y = 0, 1, \ldots,$$

where $0 < \pi < 1$. Consider testing $H_0 : \pi = \pi_0$ against $H_1 : \pi \neq \pi_0$.

(a) Write down the likelihood, $L(\pi; \underline{y})$, and hence find the generalised likelihood ratio given by $\Lambda(\underline{y}) = L(\hat{\pi}_0; \underline{y})/L(\hat{\pi}; \underline{y})$, where $\hat{\pi}_0$ is the restricted maximum likelihood estimate of π under H_0 and $\hat{\pi}$ is the maximum likelihood estimate. [9]

(b) State the critical region of the generalised likelihood ratio test in terms of $\Lambda(\underline{y})$ and explain why this only depends on the data through a sufficient statistic. [4]

(c) Use Wilks' theorem to obtain the critical region of a test with approximate significance level α for large n. [7]

End of Paper