Queen Mary
University of London

Main Examination period 2022 – May/June – Semester B

MTH6132: Relativity

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: P. Figueras, A. Shao

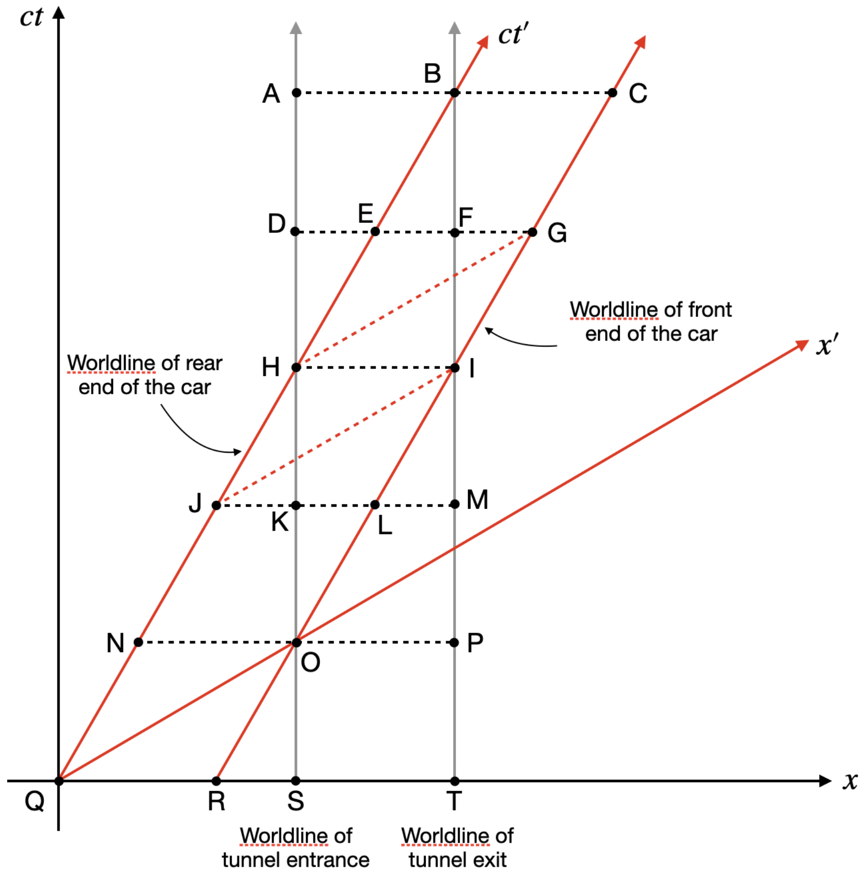

Figure 1: Spacetime diagram for Question 1. The axes (x, ct) correspond to a frame F which is at rest with respect to the tunnel while the axes (x', ct') correspond to the car's frame.

In this exam we set Newton's constant of gravitation $G = 1$ and the speed of light $c = 1$ unless otherwise stated.

Question 1 [20 marks]. A race car speeds through a tunnel at a constant velocity. The worldlines of both ends of the car and both ends of the tunnel are shown in the spacetime diagram, Fig. 1. Answer the following questions:

(a) At which of the points labeled A through T on the spacetime diagram does the front end of the car emerge from the tunnel? [4]

(b) At which point does the rear end of the car enter the tunnel? [4]

(c) At which point is the rear end of the car when the front end emerges from the tunnel in the car's frame? [4]

(d) At which point is the front end of the car when the rear end of the car enters the tunnel in the car's frame? [4]

(e) Does the car fit inside the tunnel in the tunnel's frame? How about in the car's frame? Reason your answer. [4]

Question 2 [20 marks]. A vector field ξ^a is said to be a conformal Killing vector field of the metric g_{ab} if it satisfies

$$\nabla_{(a}\xi_{b)} = \frac{1}{2}\,\psi\, g_{ab}$$

for some scalar function ψ. Recall that ξ^a is a Killing vector field if $\psi = 0$ and $\nabla_{(a}\xi_{b)} = \partial_{(a}\xi_{b)} - \Gamma^c{}_{ab}\,\xi_c$, where $\Gamma^c{}_{ab}$ are the usual Christoffel symbols.

(a) Show that this definition of conformal Killing vector field is equivalent to

$$\xi^c\,\partial_c g_{ab} + g_{ac}\,\partial_b\xi^c + g_{bc}\,\partial_a\xi^c = \psi\, g_{ab}\,.$$

[10]

(b) Using the result of Part (a), show that if ξ^a is a conformal Killing vector field of the metric g_{ab}, then ξ^a is a Killing vector field of the metric $\tilde{g}_{ab} = e^{2\phi}\, g_{ab}$, where ϕ is any function that obeys

$$2\,\xi^c\partial_c\phi + \psi = 0\,.$$

[10]

Question 3 [20 marks]. Consider the following spacetime:

$$ds^2 = -\left(1 - \frac{r^2}{\ell^2}\right)dt^2 + \frac{dr^2}{1 - \frac{r^2}{\ell^2}} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,,$$

where $\ell > 0$ is a constant.

(a) Let $u = t - \ell\,\tanh^{-1}(r/\ell)$ for $r \le \ell$. Use the coordinates (u, r, θ, ϕ) to show that the surface of $r = \ell$ is non-singular. (*Hint:* Recall that $\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}$.) **[5]**

(b) Show that the vector field $g^{ab}\partial_b u$ is null. **[5]**

(c) Show that the radial null geodesics obey either

$$\frac{du}{dr} = 0 \quad \text{or} \quad \frac{du}{dr} = -\frac{2}{1 - \frac{r^2}{\ell^2}}\,.$$

For $r < \ell$, which of these families of geodesics is outgoing, i.e., $\frac{dr}{dt} = \frac{\dot{r}}{\dot{t}} > 0$, where the dot $\dot{}$ denotes the derivative with respect to the affine parameter along the geodesics? Sketch the radial null geodesics in the (u, r) plane for $0 \le r \le \ell$, where the r-axis is horizontal and the lines of constant u are inclined at 45° with respect to the horizontal. **[10]**

Question 4 [20 marks]. Consider the following spacetime:

$$ds^2 = -\left(r^2 - \frac{2M(v)}{r}\right) dv^2 + 2\,dv\,dr + r^2(dx^2 + dy^2)\,,$$

where $M(v)$ is a function of the coordinate v.

(a) Explain in your own words the notion of black hole based on the behaviour of the null geodesics. Does this spacetime have a horizon? Reason your answer. **[5]**

(b) Use the Euler-Lagrange equations for the geodesics to show that

$$\begin{aligned}
\Gamma^{v}{}_{vv} &= r + \frac{M(v)}{r^2}\,,\\
\Gamma^{v}{}_{xx} &= -r\,,\\
\Gamma^{r}{}_{vv} &= r^3 - M(v) - \frac{2M(v)^2}{r^3} + \frac{M'(v)}{r}\,,\\
\Gamma^{r}{}_{rv} &= -\frac{1}{r}\left(r^2 + \frac{M(v)}{r}\right)\,,\\
\Gamma^{x}{}_{xr} &= \frac{1}{r}\,.
\end{aligned}$$

[5]

(c) Using the fact that, in certain units, the Ricci scalar for this geometry evaluates to $R = -12$ and the Einstein equation with a cosmological constant is given by,

$$G_{ab} - 3\,g_{ab} = 8\,\pi\,T_{ab}\,,$$

calculate the vv component of the stress tensor corresponding to the line element above. (*Hint:* recall that $\Gamma^{b}{}_{ba} = \partial_a \ln\sqrt{|g|}$.) **[10]**

Question 5 [20 marks].

(a) What does the quadrupole formula

$$\langle P \rangle = \frac{1}{5} \langle \dddot{Q}_{ij} \, \dddot{Q}^{ij} \rangle$$

compute? Reason the answer. [**5**]

(b) A point mass m undergoes a harmonic motion along the z-axis with frequency ω and amplitude L,

$$x(t) = y(t) = 0 \,, \quad z(t) = L \, \cos(\omega \, t) \,.$$

Show that the only non-vanishing component of the quadrupole moment tensor is

$$I^{zz} = m \, L^2 \, \cos^2(\omega \, t) \,.$$

[**5**]

(c) Use the quadrupole formula to compute the power radiated by the emission of gravitational waves. (*Hint:* recall that $\langle \cos(\Omega t) \rangle = \langle \sin(\Omega t) \rangle = 0$ and $\langle \cos^2(\Omega t) \rangle = \langle \sin^2(\Omega t) \rangle = \frac{1}{2}$ for a given frequency Ω.) [**10**]

End of Paper.