Queen Mary University of London

Main Examination period 2021 – May/June – Semester B
Online Alternative Assessments

MTH6132: Relativity

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

You have **24 hours** to complete and submit this assessment. When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are expected to spend about 2 hours to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: P. Figueras, J. Valiente Kroon

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In this exam G denotes Newton's gravitational constant and $\partial_a = \frac{\partial}{\partial x^a}$. Unless otherwise stated, the indices $a, b, c, \ldots$ run from 0 to 3. Also, all expressions should be simplified as much as possible.

Question 1 [15 marks]. Consider a covariant vector A_i.

(a) Write down the transformation law for A_i under general coordinate transformations. [**5**]

(b) Consider the object $F_{ij} = \partial_i A_j - \partial_j A_i$, where $\partial_i \equiv \frac{\partial}{\partial x^i}$. Compute the transformation of F_{ij} under a general coordinate transformation. Is F_{ij} a tensor? [**10**]

Question 2 [20 marks]. In this question you should assume that the dimension n of the manifold satisfies $n \geq 4$.

(a) Write the Riemann tensor as a combination of the Weyl tensor, the Ricci tensor and the Ricci scalar. [**5**]

(b) Using that $\nabla^a G_{ab} = 0$, where G_{ab} is the Einstein tensor, express the divergence of the Ricci tensor, $\nabla^a R_{ab}$, in terms of the Ricci scalar. [**5**]

(c) Using the Bianchi identity for the Riemann tensor,

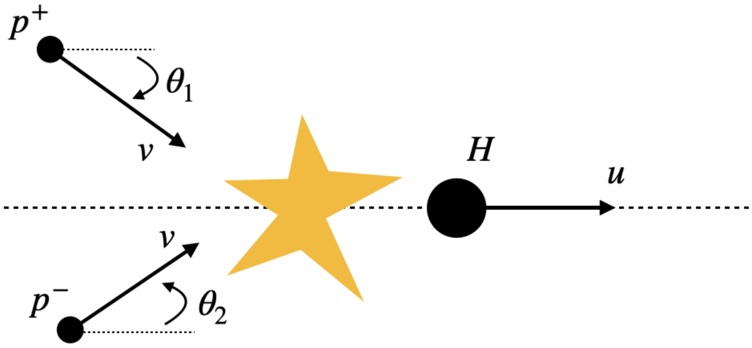
$$\nabla_{[a} R_{bc]de} = 0\,,$$

where ∇_a is the Levi-Civita connection, compute $\nabla^a C_{abcd}$. [**10**]

Question 3 [10 marks]. Consider the collision of a proton and an anti-proton both with rest mass m_p and 3-velocity v in units of $c = 1$.

(a) Assuming that the collision is head on along the x axis so that the 3-velocities of the proton and the anti-proton are v and $-v$ respectively, find the value of v so that the collision produces a Higgs boson at rest with rest mass $M_H = 125\, m_p$. [5]

(b) Consider now that both the proton and the anti-proton move along the positive x-direction forming an angle θ_1 and θ_2 respectively with respect to the x-axis in the $x - y$ plane. See the Figure below. Find v, θ_1 and θ_2 such that the Higgs boson that is formed as a result of the collision moves only along the positive x-direction with 3-velocity $u = \frac{\sqrt{3}}{2}$. [5]



Question 4 [20 marks]. Consider the general static spherically symmetric spacetime with metric

$$ds^2 = -e^{2A(r)}dt^2 + e^{2B(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,,$$

as written in the lecture notes. In addition, consider an anti-symmetric (0,2) tensor F_{ab} whose only non-vanishing components are $F_{tr} = -F_{rt} = -\frac{Q}{r^2}$, where Q is a positive constant.

(a) Show that the Einstein equations

$$G_{ab} = 8\pi G\, T_{ab}$$

are equivalent to the so called trace-reversed Einstein equations

$$R_{ab} = 8\pi G \left(T_{ab} - \frac{1}{2}\, g_{ab}\, T \right)$$

where $T \equiv T^a{}_a$ is the trace of the energy momentum tensor. [5]

(b) Compute the energy momentum tensor T_{ab} for F_{ab}:

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}\,F_b{}^c - \frac{1}{4}\,g_{ab}\,F_{cd}\,F^{cd}\right)$$

Is T_{ab} traceless? **[5]**

(c) Solve the trace-reversed Einstein equations in (a) using T_{ab} computed in (b), fixing the integration constant so that the $Q \to 0$ limit reduces to the Schwarzschild spacetime. You can use the components of the Ricci tensor for a spherically symmetric static spacetime computed in the lectures. **[10]**

Question 5 [20 marks]. Consider the line element

$$ds^2 = \left(1 - \frac{2GM}{r} + \frac{GP^2}{r^2}\right)dt^2 + \left(1 - \frac{2GM}{r} + \frac{GP^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,,$$

where M and P are constants satisfying $M > P > 0$.

(a) Find the expressions for the conserved energy E and angular momentum L along the geodesics associated to the Killing vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ respectively. **[5]**

(b) Find the effective potential $V(r)$ that governs the radial motion of the geodesics:

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \mathcal{E}\,,$$

where λ is the affine parameter along the geodesics and $\mathcal{E} = \frac{1}{2}E^2$ is defined in the lecture notes. **[5]**

(c) Sketch $V(r)$ for the null geodesics. You should assume $L > 0$. **[5]**

(d) Does this spacetime have any horizon(s)? If so, write down the spacetime metric above in ingoing Eddington-Finkelstein coordinates that are smooth at the horizon(s). **[5]**

Question 6 [15 marks]. The only non-vanishing component of the quadrupole moment tensor of a rod of length L and mass M aligned with the x'-axis is

$$I^{x'x'} = \frac{1}{12}\,M\,L^2\,.$$

(a) Find the components of the quadrupole moment tensor I^{ij} in a frame in which the rod is rotating with angular velocity Ω:

$$\begin{aligned} x &= x' \cos(\Omega\, t) + y' \sin(\Omega\, t)\,, \\ y &= -x' \sin(\Omega\, t) + y' \cos(\Omega\, t)\,, \\ z &= z' \end{aligned}$$

(Note that I^{ij} is a tensor with only spatial components and hence the indices i, j only take the values x, y and z.) [**5**]

(b) Calculate the power radiated in gravitational waves by the rotating rod. [**10**]

End of Paper.