Queen Mary
University of London

B. Sc. Examination by course unit 2015

MTH6132: Relativity

Duration: 2 hours

Date and time: 3rd June 2015, 10:00–12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Exam papers must not be removed from the examination room.

Examiner(s): Dr. J. A. Valiente Kroon

You are reminded of the following information, which you may use without proof.

- Lower case Latin indices run from 0 to 3. Repeated indices are summed over.
- The metric tensor of the Minkowski spacetime is η_{ab} such that

$$\mathrm{d}s^2 = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

- The Lorentz transformations between two frames F and F' in standard configuration are given by

$$t' = \gamma\left(t - \frac{vx}{c^2}\right), \quad x' = \gamma(x - vt), \quad y' = y, \quad z' = z$$

  where

$$\gamma = \frac{1}{\sqrt{1 - (v^2/c^2)}}$$

  and F' is moving with speed v relative to F.

- The covariant derivative of a contravariant and a covariant vector are given, respectively, by

$$\nabla_a W^b = \partial_a W^b + \Gamma^b{}_{fa} W^f, \qquad \nabla_a V_b = \partial_a V_b - \Gamma^f{}_{ba} V_f.$$

- The metric tensor satisfies:

$$g_{ab}g^{bc} = \delta_a{}^c, \quad \nabla_a g_{bc} = 0.$$

- Christoffel symbols (connection):

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}).$$

- The Riemann curvature tensor:

$$R^a{}_{bcd} = \partial_c\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bc} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}.$$

- Euler–Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^c}\right) - \frac{\partial L}{\partial x^c} = 0.$$

- Geodesic equations:

$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0.$$

Question 1.

(a) Explain the physical and/or geometrical meaning of the *Lorentz transformations*. **[2]**

(b) Explain what it means that a certain geometric object is an *invariant* with respect to the Lorentz transformations. **[2]**

(c) Show that the wave equation

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}$$

remains invariant under the Lorentz transformations given on page 2. **[8]**

Question 2.

(a) Two beams of light are shot simultaneously from the middle of a train towards its front and rear, respectively. The train is moving in a straight line with constant velocity $v < c$. For an observer moving with the train, and according to the theory of Special Relativity, which light ray reaches its destination first? Will someone standing at the station agree with this observation? Justify your answer. No computations are required. **[4]**

(b) Draw a spacetime diagram of the situation described in (a) as seen by an observer moving with the train. To draw the diagram, use coordinates for which $c = 1$. **[4]**

(c) A train travelling along the x-axis passes a platform at a speed v. A passenger at rest in the train holds a measuring ruler of length L parallel to the x-axis of the train. Determine the speed at which the train must be travelling in order for the length of the ruler as measured by the observer at the platform to be $L/3$. **[4]**

Question 3.

(a) Give the transformation formula of a (1,3)-tensor $S^a{}_{bcd}$ and show that $S_{bd} = S^a{}_{bad}$ is a $(0, 2)$-tensor. **[5]**

(b) Explain the geometrical meaning of the *metric tensor* g_{ab}. **[2]**

(c) The metric tensor g_{ab} is related to the line element $\mathrm{d}s^2$ through the expression

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b.$$

Use the invariance of this expression to show that g_{ab} is a $(0, 2)$-tensor. **[5]**

Question 4.

(a) Let B_a and W_{ab} denote, respectively, a covariant vector and covariant tensor of rank 2. Is the expression

$$W_{ab} = \partial_b V_a$$

a valid tensorial expression? Justify. **[3]**

(b) The Riemann curvature tensor $R^a{}_{bcd}$ is defined in terms of the covariant derivatives of a vector V^a through the relation

$$\nabla_a \nabla_b V^c - \nabla_b \nabla_a V^c = R^c{}_{dab} V^d.$$

Use the above formula to explain why the Riemann tensor is a $(1,3)$-tensor. **[3]**

(c) Explain what is understood by a solution to the vacuum Einstein field equations. **[3]**

(d) Is the metric given by the line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + 2\mathrm{d}x^2 + 3\mathrm{d}y^2 + 4\mathrm{d}z^2$$

a solution to the vacuum Einstein field equations? Explain your answer. **[3]**

Question 5. The line element of the metric of the 2-dimensional sphere is given in spherical coordinates (θ, φ) by

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2.$$

(a) Calculate the covariant and contravariant components of the metric tensor for this line element. **[2]**

(b) Compute, by the method you prefer, all the Christoffel symbols for this metric. **[8]**

(c) Using the formula for the Riemann tensor given in page 2, show that

$$R^1{}_{212} = \sin^2\theta.$$

[3]

(d) Use the result in (c) and the symmetries of the Riemann tensor to compute the component $R^1{}_{221}$. **[1]**

Question 6. In this question consider units for which $c = 1$.

(a) Define the notion of *proper time* in Special Relativity. Explain the meaning of this concept. **[2]**

(b) Define the *4-velocity* $\bar{u}$ of a timelike particle in Special Relativity. Show that for such a particle

$$|\bar{u}|^2 = -1.$$

Define the *4-momentum* $\bar{p}$ of a timelike particle. Prove that

$$|\bar{p}|^2 = -m_0^2$$

where m_0 is the rest mass of the particle. **[6]**

(c) Show that the 4-momentum can be expressed in terms of the 3-velocity of the particle, $\underline{v}$, and its rest mass as

$$\bar{p} = m_0 \gamma(v)(1, \underline{v})$$

where

$$\gamma(v) = \frac{1}{\sqrt{1-v^2}}, \qquad v = |\underline{v}|.$$

[6]

Question 7. An atom of rest mass m_0 at rest in a laboratory absorbs a photon of frequency ν. Use the conservation law of 4-momentum to find the velocity and rest mass of the resulting atom. **[10]**

Question 8. Consider the Schwarzschild line element

$$\mathrm{d}s^2 = -\left(1 - \frac{2Gm}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2Gm}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)$$

(a) What physical situation is described by the metric associated to this line element? What situation is described by $m = 0$? **[3]**

(b) What can you say about the geometry at $r = 2Gm$? What about at $r = 0$? **[2]**

(c) Use the line element to derive the equation expressing that the 4-velocity of a photon moving on the Schwarzschild spacetime has zero norm. **[3]**

(d) Let $\bar{t}$ denote a new time coordinate given by

$$\bar{t} = t + 2Gm \ln|r - 2Gm|.$$

Show that, in terms of this coordinate, the Schwarzschild metric takes the form

$$\mathrm{d}s^2 = -\left(1 - \frac{2Gm}{r}\right)\mathrm{d}\bar{t}^2 + \frac{4Gm}{r}\mathrm{d}\bar{t}\mathrm{d}r + \left(1 + \frac{2Gm}{r}\right)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right).$$

[6]

End of Paper.