MTH6128 / MTH6128P: Number Theory

Duration: 2 hours

Date and time: 2 June 2016, 10:00 to 12:00

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are **not** permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): X. Li

Turn Over

Question 1.

(a) Define the terms

    (i) **algebraic number**;

    (ii) **algebraic integer**;   **[4]**

    (iii) **transcendental number**.

(b) Which of the following numbers are algebraic integers? Explain, stating explicitly which theorems you use.

    (i) $\dfrac{5 + \sqrt{15}}{2}$;   **[5]**

    (ii) $\dfrac{1}{2}\sqrt{41} - \dfrac{3}{2}$.

(c) What does it mean for a positive irrational number to be *approximable by rationals to order m*?   **[3]**

(d) Is $\sqrt[3]{2}$ approximable by rationals to order 4? Justify your answer. State explicitly which theorems you use in the proof.   **[6]**

Question 2.

(a) Find the continued fraction for $\dfrac{7 + \sqrt{11}}{3}$.   **[8]**

(b) Calculate the value of the infinite continued fraction $[2; \overline{4, 1}]$.   **[8]**

Question 3.

(a) You are given that
$$[8; \overline{1, 1, 5, 5, 1, 1, 16}]$$
is the continued fraction for $\sqrt{73}$. Using this, find positive integers x and y such that $x^2 + y^2 = 73$.   **[6]**

(b) You are given that
$$[8; \overline{1, 1, 1, 16}]$$
is the continued fraction for $\sqrt{75}$. Using this, find all the integer solutions of the equation
$$x^2 - 75y^2 = \pm 1.$$
Explain why you have found ALL the integer solutions.   **[8]**

(c) Let n be a positive integer which is not a square. Suppose that x, y and x', y' are positive integers satisfying $x^2 - ny^2 = \pm 1$ and $(x')^2 - n(y')^2 = \pm 1$. Assume that $x < x'$.

Show that $y < y'$. State explicitly which theorems you use in the proof.   **[8]**

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Question 4.

(a) Let p be a prime. What is a **primitive root** (mod p)? **[3]**

(b) Find a primitive root (mod 11). **[5]**

(c) Find an integer n with $n^5 \equiv 1$ (mod 11) and $n^4 \not\equiv 1$ (mod 11). Show that the integer you have found has the required properties. **[3]**

(d) Does there exist an integer n with $n^3 \equiv 1$ (mod 11) and $n^2 \not\equiv 1$ (mod 11)? Justify your answer. State explicitly which theorems you use in the proof. **[5]**

Question 5.

(a) Let p be an odd prime. What is a **quadratic residue** (mod p)? **[2]**

(b) Let p be an odd prime, and let a be an integer. Define the **Legendre symbol** $\left(\dfrac{a}{p}\right)$. **[3]**

(c) Calculate the value of $\left(\dfrac{18}{71}\right)$. You should state clearly any rules for computing Legendre symbols that you use, but are not required to prove them. **[6]**

(d) Let p be an odd prime. Show that we have **[5]**

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} = \begin{cases} +1 & \text{if } p \equiv 1 \ (\text{mod } 4), \\ -1 & \text{if } p \equiv 3 \ (\text{mod } 4). \end{cases}$$

Question 6.

(a) What is a **quadratic form** over the integers? **[2]**

(b) Give an example of a quadratic form which is indefinite. Explain why the example you gave has the desired property. **[2]**

(c) Find a reduced positive definite quadratic form which is equivalent to **[4]**

$$3x^2 + 2xy + y^2.$$

(d) What is meant by saying that an integer is *represented* by a quadratic form? What can we say about the integers represented by two equivalent quadratic forms? **[4]**

End of Paper.