Main Examination period 2017

MTH6121 / MTH6121P: Introduction to Mathematical Finance

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: N. Rodosthenous, A. Baule

In this exam all interest rates quoted are annual interest rates and all continuous times are measured in years, unless otherwise stated. Also, all expressions should be simplified as much as possible.

Question 1. [25 marks]

(a) Suppose that a bank charges a daily compounded (annual) interest rate of 18% for every day someone delays a scheduled payment of a loan. How much will you have to pay if you delay a payment of £500 for 2 weeks? [**3**]

(b) Consider a 5-year contract that will pay its holder £100 at the end of each of the following 5 years with a final additional payment of £10,000 at the end of the fifth year. Assuming that the interest rate (compounded annually) will remain 1% during the life of this contract, find the price V of the contract (present value of the cash flow generated by this contract). [**5**]

(c) Assume in the contract of part (b) that the interest rate increases, will the present value V increase or decrease? Justify your answer. [**2**]

(d) An investor who is interested in buying the contract of part (b) but cannot currently pay the value V, suggests to pay now £5,000 and an amount A at the end of the sixth year. Assuming a constant interest rate of 1%, how much should the amount A be? You may leave your answer in terms of V. [**3**]

(e) Suppose now that an investor owns shares of company Alpha. Given recent bad news, the probability that the share price of company Alpha decreases this month is assumed to be 70%, but since the effect is only short term, the probability it decreases next month is assumed to be 50%. Also, the probability it decreases both this and next month is assumed to be 40%. The investor is interested in knowing what is the probability that the share price does not decrease too much. In particular, what is the probability that

  (i) it decreases this month but does not decrease next month? [**3**]

  (ii) it does not decrease next month, given that it decreases this month? [**3**]

  (iii) it decreases this or next month, but does not decrease in both months? [**3**]

(f) Let A be the event that the share price of company Alpha decreases this month and B the event that it decreases next month. Are A and B independent? Justify your answer. [**3**]

Question 2. [25 marks]

(a) Explain what is meant by a lognormally distributed random variable Y with parameters μ and σ. What values can the parameters take? [**3**]

(b) Suppose that $S(n)$ denotes the price of the share of company Epsilon at the end of the n-th day, for $n \in \mathbb{N}$, which is given by the IID lognormal model. State the assumptions of this model regarding the sequence

$$Y_n = \frac{S(n)}{S(n-1)}, \quad \text{for } n \in \mathbb{N}.$$ [**3**]

(c) Consider the sequence $\big(Y_n\big)_{n\in\mathbb{N}}$ defined in part (b). Show that

$$\prod_{i=1}^{n} Y_i^2 \sim \text{LogNormal}(m_n, s_n^2)$$

for appropriate parameters m_n and s_n. Give explicit formulae for m_n and s_n. Here, $\prod_{i=1}^{n} Y_i^{a_i}$ denotes the product $Y_1^{a_1} Y_2^{a_2} \cdots Y_n^{a_n}$. [**5**]

(d) Prove the following result.

Lemma *If $S(n)$ is given by the IID lognormal model, then for any $0 < k < n$, we have*

$$\frac{S(n)}{S(k)} \sim \text{LogNormal}\big((n-k)\mu, (n-k)\sigma^2\big) \quad \textit{for any } n \in \mathbb{N}.$$

Hint: You may use the expressions of Y_n defined in part (b). [**6**]

(e) Consider the daily share price $S(n)$ of company Epsilon from part (b), with $\mu = 0.02$ and $\sigma^2 = (0.136)^2$. Determine the probability that the share price is higher at the end of the 21st day than at the end of the 7th day.
Hint: You may use the Lemma from part (d). [**5**]

(f) What is the probability that the price is higher at the end of the 35th day than at the end of 21st day and the later is higher than the price at the end of 7th day? [**3**]
Hint: You may use part (e).

Question 3. [25 marks] Recall that the “Arbitrage theorem” provides a result for investors who bet on n wagers based on an experiment with m possible outcomes and return functions $r_i(j)$ for $i = 1,...,n$ and $j = 1,...,m$.

Theorem *Exactly one of the following two statements (I) and (II) is true:*
(I) Either there exists a probability vector $\mathbf{p} = (p_1, p_2, ..., p_m)$ *such that*

$$\sum_{j=1}^{m} p_j r_i(j) = 0 \ \ for\ all\ i = 1,...,n. \tag{1}$$

(II) Or there exists a betting strategy $\mathbf{x} = (x_1, x_2, ..., x_n)$ *for which*

$$\sum_{i=1}^{n} x_i r_i(j) > 0 \ \ for\ all\ \ j = 1,...,m. \tag{2}$$

(a) Prove that if we assume that both (1) and (2) hold true for all i and j, $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, we arrive at a contradiction. **[6]**

(b) Explain in words what a European put option is. **[5]**

(c) Assume that the interest rate is 1% per time period. Suppose that at time 0, company Beta’s share is traded at the price £55, and that at time 1, your financial analyst predicts that it will be traded either at £51 or £58. Consider a European put option on this share with maturity time $T = 1$ and strike price $K = £56$. What is the no-arbitrage price P of this put option?
You should use the Arbitrage Theorem with two wagers; purchase one share and purchase one put option. **[7]**

(d) Suppose now that you are also interested in the share of company Gamma, which is currently traded at the price £20. Your financial analyst predicts that over the next 3 time periods the share price will follow the multi-period Binomial model and can either move upwards with a factor $u = 1.05$ or downwards with a factor $d = 0.9$. Draw the Binomial tree with the evolution of the share in the next 3 time periods and its possible future states. **[3]**

(e) Assume that the interest rate is 1% per time period. What is the no-arbitrage price of a European put option written on the share of company Gamma with maturity time $T = 3$ and strike price $K = £18$?
You may use the Binomial model formula without proof, provided that you verify the necessary conditions for its application. **[4]**

Question 4. [25 marks]

(a) What is a *Brownian motion* or a *Wiener process*? **[3]**

(b) Let $W(t)$ be a Brownian motion. Then, consider the Brownian motion with drift given by $Y(t) = 4 + 0.2t + 0.4W(t)$. Determine $\mathbb{E}[Y(1) - Y(2)]$. **[2]**

(c) Determine $\mathrm{Cov}\big(Y(1), Y(2)\big)$. **[3]**

(d) Determine $\mathbb{V}\mathrm{ar}\big(Y(1) - Y(2)\big)$ **[3]**

(e) Consider the share of company Delta whose price $S(t)$ at time t is given by a geometric Brownian motion with drift parameter μ and volatility parameter σ. What is the probability that the share price exceeds the level K at time T? Your answer should be given in terms of the cumulative distribution function Φ of a standard normal random variable. **[5]**

(f) Suppose now that the share price $S(t)$ of company Delta at time t evolves according to the Black and Scholes model with volatility parameter σ in a market with a continuously compounded interest rate r. Write down the expression for $S(t)$. **[2]**

(g) Recall that the Black and Scholes price C of a European call option with maturity T and strike price K is given by the formula

$$C = S\Phi(\omega) - Ke^{-rT}\Phi(\omega - \sigma\sqrt{T}), \tag{3}$$

where

$$\omega = \frac{rT + \frac{\sigma^2 T}{2} - \log\frac{K}{S}}{\sigma\sqrt{T}}. \tag{4}$$

Prove using the definition (4) of ω that the equation

$$Ke^{-rT}\Phi'(\omega - \sigma\sqrt{T}) = S\Phi'(\omega) \tag{5}$$

holds true. **[3]**

(h) What is the rate with which the call option price C given by (3) changes, if the current interest rate r changes and the rest of the model parameters S, σ, K, T are kept constant? Give all details of the derivation of the desired result.
Hint: You may use the equation (5). **[4]**

End of Paper—An appendix of 1 page follows.

Table of the cumulative standard normal distribution

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2}\,\mathrm{d}t.$$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

End of Appendix.