MTH6109 / MTH6109P: Combinatorics

Duration: 2 hours

Date and time: 24 May 2016, 14:30h–16:30h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): R. A. Wilson

Question 1. Suppose you have 24 identical sweets to distribute to five children.

(a) How many ways are there of distributing the sweets to the children? [**4**]

(b) How many ways are there if every child gets at least one sweet? [**4**]

(c) Now suppose there are 12 red sweets and 12 green sweets. In how many ways can the sweets be distributed so that each child gets at least one sweet of each colour? [**4**]

(d) Now suppose that two of the children will not eat green sweets and one other will not eat red sweets. How many ways are there of distributing 12 red and 12 green sweets in accordance with these restrictions? [**4**]

[Justify your answers. You need not evaluate binomial coefficients in your answers.]

Question 2. Let $A = \{1, 2, \dots, n\}$ and $B = \{1, 2, \dots, k\}$, where k, n are positive integers.

(a) Derive a formula for the number of functions $f : A \to B$. [**4**]

(b) Derive a formula for the number of injective (one-to-one) functions $f : A \to B$. [**4**]

(c) State the Principle of Inclusion and Exclusion. [**4**]

(d) Derive a formula for the number of surjective (onto) functions $f : A \to B$, in the case when $k = 4$ and $n = 6$. [You need not evaluate the answer.] [**6**]

Question 3. (a) Solve the recurrence relation

$$a_n = 2a_{n-1} + 15a_{n-2}$$

with initial conditions $a_0 = 3$, $a_1 = -1$. [**4**]

(b) Solve the recurrence relation

$$b_n = 3b_{n-1} - 3b_{n-2} + b_{n-3}$$

with initial conditions $b_0 = 1$, $b_1 = 3$, $b_2 = 9$. [**6**]

(c) Suppose that $c_0 = c_1 = 1$ and

$$c_n = c_{n-1} + (n-1)c_{n-2}.$$

Show that $c_n \geq \sqrt{n!}$, for all $n \geq 0$. [**6**]

Question 4. Let C_n denote the number of ways of bracketing the expression

$$a_1 a_2 a_3 \cdots a_{n-1} a_n,$$

and let

$$C(x) = \sum_{n=1}^{\infty} C_n x^{n-1} = C_1 + C_2 x + C_3 x^2 + \cdots.$$

(a) Show that

$$C_n = \sum_{i=1}^{n-1} C_i C_{n-i},$$

with $C_1 = C_2 = 1$. [**4**]

(b) Hence or otherwise compute C_4. [**2**]

(c) Show that $xC(x)^2 = C(x) - 1$. [**4**]

(d) Hence or otherwise find a closed formula for $C(x)$. [**4**]

Question 5. (a) Define the terms **bipartite**, **matching** and **neighbourhood**, as used in graph theory. [**4**]

(b) State Hall's Theorem about the existence of matchings in bipartite graphs. [**4**]

(c) Define the term **Latin rectangle**, and give an example of a 5×3 Latin rectangle. [**4**]

(d) Prove that if $k < n$ then every $n \times k$ Latin rectangle can be extended to an $n \times (k+1)$ Latin rectangle.

[You may assume that Hall's condition is satisfied in any r-regular bipartite graph.]

Illustrate by extending your example in (c) to a 5×4 Latin rectangle. [**6**]

Question 6. (a) State Euler's equation, for a connected plane graph with n vertices, p edges, and q faces. [**2**]

(b) Deduce that, provided $n \geq 3$,

$$p \leq 3n - 6.$$

[**6**]

(c) Hence show that the average degree of the vertices in any planar graph is strictly less than 6. [**4**]

(d) State and prove the five-colour theorem. [**6**]

End of Paper.