Queen Mary
University of London

B. Sc. Examination by course unit 2015

MTH6107: Chaos & Fractals

Duration: 2 hours

Date and time: 13th May 2015, 14.30–16.30

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You should attempt ALL questions. Marks awarded are shown next to the questions.

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Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Examiner(s): O. Jenkinson

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Question 1. (a) [4 marks] For a map $f : \Sigma \to \Sigma$ on a non-empty set Σ, what does it mean to say that $x \in \Sigma$ is a *periodic point* for f, and how is its *minimal period* defined?

(b) [6 marks] Give a detailed statement of Sharkovsky's Theorem.

(c) [6 marks] Order the integers from 1 to 25 inclusive using Sharkovsky's ordering.

(d) [4 marks] For the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x-1)(1-3x^2/2)$, determine the orbit of the point 0.

(e) [4 marks] Show that the map f of part (d) above has a point of minimal period n for every $n \in \mathbb{N}$.

Question 2. Suppose the map $f : [0,1] \to [0,1]$ is defined by

$$f(x) = \begin{cases} 8x^3 & \text{if } x \in [0,1/2] \\ 2(1-x) & \text{if } x \in (1/2,1] \, . \end{cases}$$

(a) [6 marks] Determine the three fixed points of f.

(b) [6 marks] Compute the multiplier of each fixed point, and use this to determine whether the point is unstable, stable, or superstable.

(c) [5 marks] For $x = 2/5$, compute the points $f(x)$, $f^2(x)$, and $f^3(x)$.

Describe, with justification, the behaviour of $f^n(x)$ as $n \to \infty$.

Question 3. (a) [9 marks] Define what it means for $f : \mathbb{R} \to \mathbb{R}$ to be

(i) a *homeomorphism*,

(ii) a *diffeomorphism*,

(iii) *order reversing*.

(b) [10 marks] Prove that an order reversing diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ has precisely one fixed point.

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Question 4. Let $\mathcal{H}$ denote the collection of compact subsets of $\mathbb{R}$. Let $\Phi : \mathcal{H} \to \mathcal{H}$ be the iterated function system defined by the two maps $\phi_1(x) = x/10$ and $\phi_2(x) = (x+3)/10$, and let C_k denote $\Phi^k([0,1])$ for $k \geq 0$.

(a) [5 marks] For $A, B \in \mathcal{H}$, how is the *Hausdorff distance* $h(A, B)$ defined?

(b) [4 marks] Write down the sets C_1 and C_2.

(c) [5 marks] Compute $h(C_1, C_2)$.

(d) [3 marks] If C_k is expressed as a disjoint union of N_k closed intervals, compute the number N_k.

(e) [3 marks] What is the common length of each of the N_k closed intervals whose disjoint union equals C_k?

(f) [5 marks] Given a set $A \subset \mathbb{R}$, how is its *box dimension* defined?

(g) [5 marks] Using your answers to parts (d) and (e), or otherwise, show that if the box dimension of $C = \cap_{k=0}^{\infty} C_k$ exists then it must equal $\log 2 / \log 10$.

(h) [5 marks] Give a description of the members of C in terms of the digits of their decimal expansion.

(i) [5 marks] If $f : C \to C$ is defined by $f(x) = 10x \pmod 1$ then find a point $x \in C$ which has minimal period 3 under f.

End of Paper.

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