Main Examination period 2017

MTH6104 / MTH6104P: Algebraic structures II

Duration: 2 hours

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Examiners: Matthew Fayers and Leonard Soicher

In this paper, we use the following notation.

- $\mathcal{C}_n$ denotes the cyclic group of order n.
- $\mathcal{U}_n$ is the set of integers between 0 and n which are prime to n, with the group operation being multiplication modulo n.
- $\mathcal{D}_{2n}$ is the group with $2n$ elements

$$1,\ r,\ r^2,\ \dots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \dots,\ r^{n-1}s.$$

  The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.
- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \dots, n\}$ (with the group operation being composition).
- If p is a prime, then $\mathbb{Z}/p\mathbb{Z}$ is the set $\{0, 1, \dots, p-1\}$, with addition and multiplication modulo p. $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ is the group of 2×2 matrices with entries in $\mathbb{Z}/p\mathbb{Z}$ and with determinant 1, with the group operation being matrix multiplication.

Question 1. [25 marks]

(a) Give the definition of a **group**. **[3]**

(b) Prove that the identity element in a group is unique. **[4]**

(c) Suppose F is a set consisting of five elements a, b, c, d, e, and a binary operation is defined on F by the following table.

	a	b	c	d	e
a	a	b	c	d	e
b	b	e	d	a	c
c	c	a	e	b	d
d	d	c	b	e	a
e	e	d	a	c	b

Is F a group under this operation? Justify your answer. **[5]**

Suppose G is a group and $g \in G$.

(d) How are the **powers** g^n defined, for $n \in \mathbb{Z}$? Give the definition of the **order** $\mathrm{ord}(g)$ of g. **[5]**

(e) Suppose $\mathrm{ord}(g)$ is even. What is $\mathrm{ord}(g^2)$? Justify your answer. **[4]**

(f) Suppose $\mathrm{ord}(g)$ is odd. What is $\mathrm{ord}(g^2)$? Justify your answer. **[4]**

[You may use standard rules for manipulating powers of elements.]

Question 2. [25 marks] Write an essay on conjugacy, centres and centralisers. *[You should include precise definitions and statements of results, illustrated by examples, and give some proofs.]*

Question 3. [25 marks]

(a) Suppose G is a group. Define what it means to say that G is **simple**. *[You do not need to define what a normal subgroup is.]* **[2]**

Suppose $g \in \mathcal{S}_n$.

(b) Explain how to write g in **disjoint cycle notation**. **[3]**

(c) Prove that $\operatorname{ord}(g)$ is the least common multiple of the lengths of the cycles of g when g is written in disjoint cycle notation. **[4]**

(d) Give the definition of a **transposition** in $\mathcal{S}_n$. Write the permutation $(1\ 6\ 7)(2\ 5\ 3\ 4)$ as a product of transpositions. **[5]**

(e) Give the definition of the **alternating group** $\mathcal{A}_n$. **[2]**

(f) Prove that any element of $\mathcal{A}_n$ can be written as a product of 3-cycles. **[5]**

(g) Write down two further results on 3-cycles in $\mathcal{A}_n$ which are used with part (f) to show that $\mathcal{A}_n$ is simple for $n \geqslant 5$. **[4]**

Question 4. [25 marks] Suppose G is a group and X is a set.

(a) Define what is meant by an **action** of G on X. **[3]**

(b) Give two examples of actions of $\mathcal{D}_8$ on itself, one of which is transitive, and the other not transitive. *[You should say clearly how the actions are defined and which one is transitive, but you do not need to prove anything.]* **[4]**

(c) Suppose π is an action of G on X, and $x \in X$. Define what is meant by the **stabiliser** of x, and prove that it is a subgroup of G. **[6]**

(d) Give a precise statement of the Orbit–Stabiliser Theorem. **[3]**

Now let p be a prime, and $G = \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$. Let X be the set of non-zero column vectors of length 2 with entries in $\mathbb{Z}/p\mathbb{Z}$, and let G act on X by $\pi_g(x) = gx$.

(e) By considering the orbit of the vector $\begin{pmatrix}1\\0\end{pmatrix}$, prove that this action is transitive. **[5]**

(f) Hence use the Orbit–Stabiliser Theorem to find $|G|$. **[4]**

Question 5. [25 marks] Suppose G is a group and H is a subgroup of G.

(a) Suppose $g \in G$. Define what is meant by the **right coset** Hg, and the **index** of H in G. **[4]**

(b) Give a precise statement of Lagrange's Theorem. **[3]**

(c) Now let $K = \{1, 9, 25\}$. Find all the right cosets of K in $\mathcal{U}_{28}$. *[You may assume that K is a subgroup of $\mathcal{U}_{28}$.]* **[4]**

(d) Prove that if H has index 2 in G, then H is a normal subgroup of G. *[You may assume that every element of G lies in exactly one left coset of G, and similarly for right cosets. You may also assume that if $gH = Hg$ for every $g \in G$, then H is a normal subgroup.]* **[5]**

Now suppose p is a prime number and G is finite.

(e) Give the definition of a **Sylow p-subgroup** of G. **[2]**

(f) Give a precise statement of Sylow's Theorem 1. **[3]**

(g) Use part (d) and Sylow's Theorem 1 to show that there is no simple group of order 50. **[4]**

Question 6. [25 marks] Suppose G and H are groups.

(a) Give the definitions of the following:

- a **homomorphism** from G to H;
- an **isomorphism** from G to H;
- an **automorphism** of G. **[6]**

(b) Suppose $\phi : G \to H$ is a homomorphism, let $N = \ker(\phi)$ and suppose $K \leqslant G$. Prove that $\phi^{-1}(\phi(K)) = NK$. **[5]**

(c) Give a precise statement of the Correspondence Theorem. **[3]**

(d) Let $G = \mathcal{C}_{40}$. Find all the subgroups of G, and draw a diagram showing which subgroups contain which others. *[You do not need to prove anything.]* **[3]**

(e) Let $\phi : \mathcal{C}_{40} \to \mathcal{C}_{40}$ be the homomorphism which sends g to g^{10} for every $g \in \mathcal{C}_{40}$. Find $\mathrm{im}(\phi)$ and $\ker(\phi)$, and show how subgroups correspond under the Correspondence Theorem. *[You do not need to prove anything.]* **[5]**

(f) Give an example of an outer automorphism ϕ of $\mathcal{D}_8$ which satisfies $\phi(r) \neq r$. *[You do not have to prove anything, but you should say what $\phi(g)$ is for each $g \in \mathcal{D}_8$.]* **[3]**

End of Paper.