Queen Mary
University of London

Main Examination period 2018

MTH5123: Differential Equations

Duration: 2 hours

Student number										**Desk number**			

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Write your solutions in the spaces provided in this exam paper. If you need more paper, ask an invigilator for an additional booklet and attach it to this paper at the end of the exam.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiners: Weini Huang, Oliver Jenkinson

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Question	Mark	Comments
1	/ 20	
2	/ 30	
3	/ 25	
4	/ 25	
Total		

Question 1. [20 marks]

(a) Find the solution to the Initial Value Problem of the first-order ordinary differential equation (ODE)

$$y' = -y/(1+x), \;\; y(1) = 2.$$

[6]

(b) Find the general solution to the homogeneous second-order linear ODE

$$y'' + y' - 12y = 0.$$

[6]

(c) Use the solution in (b) to find the general solution to the inhomogeneous second-order linear ODE

$$y'' + y' - 12y = -3e^{-x}.$$

[8]

Write your solutions here

Continue-1: solutions to question 1

Continue-2: solutions to question 1

Question 2. [30 marks]

Consider the differential equation

$$(1 - by\sin(x)) + 2\cos(x)y' = 0.$$

(a) Find the value of the parameter b for which the given differential equation is exact. **[8]**

(b) For the parameter b found in (a) above, find the solution which satisfies the initial condition $y(0) = 1$. **[12]**

(c) Consider the initial value problem

$$y' = f(x,y), \quad f(x,y) = \sqrt{3y^2 + 16}, \quad y(1) = 0\,.$$

Show that the Picard-Lindelöf Theorem ensures the uniqueness and existence of a solution to the above problem in a rectangular domain $|x-a| \leqslant A$, $|y-b| \leqslant B$, and specify the parameters a and b. Write down the maximal value of the width A for $B = 4$. **[10]**

Continue-1: solutions to question 2

Continue-2: solutions to question 2

Question 3. [25 marks]

Write down the solution to the following Boundary Value Problem (BVP) for the second order inhomogeneous differential equation

$$2x^2y'' - 4y = f(x), \quad y(2) = 0, \ y'(3) = 0$$

by using the Green's function method along the following lines:

(a) Using that the left-hand side of the ODE is in the form of an Euler-type equation determine the general solution of the corresponding homogeneous ODE. **[8]**

(b) Formulate the corresponding left-end and right-end initial value problems and use their solutions to construct the Green's function $G(x, s)$. Further represent the solution to the BVP in terms of the found $G(x, s)$ for the particular choice $f(x) = e^x$ (you do not need to evaluate the resulting integrals). **[17]**

Write your solutions here

Continue-1: solutions to question 3

Continue-2: solutions to question 3

Question 4. [25 marks]

Consider a system of two nonlinear first-order ODEs where x and y are functions of independent variable t :

$$\dot{x} = -x - 3y - 3x^3, \quad \dot{y} = \frac{4}{3}x - y - \frac{1}{3}x^3$$

(a) Write down in matrix form the system obtained by linearization of the above equations around the point $x = y = 0$ and find the corresponding eigenvalues and eigenvectors. **[8]**

(b) Write down the general solution of the linear system. Discuss the stability of the zero solution of such a linear system and determine the value $\lim_{t\to\infty} x(t)$. **[4]**

(c) Find the solution of the linear system corresponding to the initial conditions $x(0) = 2$, $y(0) = 0$. Determine the type of equilibrium for the system and describe in words the shape of the trajectory in the phase plane corresponding to the specified initial conditions. Determine the tangent vector to the trajectory at $t = 0$. **[8]**

(d) Demonstrate how to use the function $V(x, y) = \frac{4}{3}x^2 + 3y^2$ to investigate the stability of the full non-linear system. **[5]**

Write your solutions here

Continue-1: solutions to question 4

Continue-2: solutions to question 4

End of Paper – An appendix of 2 pages follows.

Formula Sheet

Integrations

$$\int \frac{dx}{x+1} = \ln|x+1| + C$$

The roots of the characteristic equations of the homogeneous 2nd-order linear ODE,

$$a_2\lambda^2 + a_1\lambda + a_0 = 0,$$

are $\lambda_1 = \frac{-a_1+\sqrt{a_1^2-4a_2a_0}}{2a_2}$ and $\lambda_2 = \frac{-a_1-\sqrt{a_1^2-4a_2a_0}}{2a_2}$.

Reminder on ODEs:

Exact first-order ODEs

If the equation

$$P(x,y) + Q(x,y)\,\frac{dy}{dx} = 0$$

is exact, its solution can be found in the form $F(x,y) = Const.$ where

$$P = \frac{\partial F}{\partial x} \quad \text{and} \quad Q = \frac{\partial F}{\partial y}$$

Picard-Lindelöf Theorem for the existence and uniqueness of the solution to the initial value problems

For the initial value problem

$$\frac{dy}{dx} = f(x,y), \quad y(a) = b$$

the Picard-Lindelöf Theorem guarantees the uniqueness and existence of the solution to the above problem in a rectangular domain $\mathcal{D} = (|x-a| \leqslant A,\ |y-b| \leqslant B)$ in the xy plane, provided
(i) $f(x,y)$ is continuous and therefore bounded in $\mathcal{D}$;
(ii) the parameters A and B satisfy $A \leq B/M$ where $M = max_{\mathcal{D}}|f(x,y)|$;
(iii) $|\frac{\partial f}{\partial y}|$ is bounded in $\mathcal{D}$.

Boundary value problems of second-order linear ODEs and the Green's function method

If there exists a unique solution $y(x)$ to a non-homogeneous **boundary value problem** for ODE

$$\mathcal{L}(y) = a_2(x)y'' + a_1(x)y' + a_0(x) = f(x)$$

in an interval $x \in [x_1, x_2]$ with linear homogeneous boundary conditions.

$$\alpha y'(x_1) + \beta y(x_1) = 0, \quad \gamma y'(x_2) + \delta y(x_2) = 0$$

it can be found by the Green's function method:

$$y(x) = \int_{x_1}^{x_2} G(x,s)\, f(s)\, ds, \quad G(x,s)) \equiv \begin{cases} A(s)\, y_L(x), & x_1 \leqslant x \leqslant s \\ B(s)\, y_R(x), & s \leqslant x \leqslant x_2 \end{cases}$$

where

$$A(s) = \frac{y_R(s)}{a_2(s)W(s)}, \quad B(s) = \frac{y_L(s)}{a_2(s)W(s)}, \quad W(s)) \equiv y_L(s)y_R'(s) - y_R(s)y_L'(s)$$

and $y_L(x), y_R(x)$ are solutions to the left/right initial value problems:

$$\mathcal{L}(y) = 0,\ y(x_1) = \alpha,\ y'(x_1) = -\beta; \quad \text{and} \quad \mathcal{L}(y) = 0,\ y(x_2) = \gamma,\ y'(x_2) = -\delta$$

The orbital derivative for a Lyapunov function $V(x,y)$ is defined as:

$$\mathcal{D}_f V = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$$

End of Appendix.