Queen Mary University of London

Examination period 2018

MTH5121: Probability Models

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: I. Goldsheid, D. Ellis

Question 1. [26 marks] Suppose that X and Y are two random variables.

(a) State the definition of independence of two random variables X and Y. **[3]**

(b) Suppose that the joint density function $f_{X,Y}$ is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{4}y & \text{if } 0 < y < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Find the marginal density functions $f_X(x)$ and $f_Y(y)$. **[8]**

(ii) Find the probability $\mathbb{P}(X \in [0,\,1] \text{ and } Y \in [0,\,1])$. **[6]**

(iii) Are X and Y independent? Justify your answer. **[3]**

(iv) Find the conditional density function $f_{Y|X=x}(y)$ and compute $\mathbb{E}(Y^3|X)$. **[6]**

Question 2. [28 marks] A random walk on a line starts from n, where $M \leqslant n \leqslant N$. The probability of a jump to the right is p and the probability of a jump to the left is $q = 1 - p$. The walk stops once it reaches M or N.

(a) Suppose that $p = 2/5$, $q = 3/5$, the random walk starts from position 0, and $N = 3$ (in other words, the walk is on $(-\infty,\,3])$. What is the probability that this walk reaches 3? **[10]**

Hint. You may use, without proof, the formula for r_n, the probability that the walk starting from n reaches N before M.

(b) Suppose that a random walk is starting from n, $0 \leqslant n \leqslant N$. Let T_n be the time the walk takes to reach 0 or N and E_n be the expected duration of the walk (that is $E_n = \mathbb{E}(T_n)$). Prove that

$$\begin{aligned} &E_0 = E_N = 0, \\ &E_n = pE_{n+1} + qE_{n-1} + 1 \qquad \text{for } 0 < n < N. \end{aligned}$$

[15]

(c) Write down (do not prove) the formula for E_n in the case $p = q = \frac{1}{2}$. **[3]**

Question 3. [**18 marks**] Let $Y_0 = 1, Y_1, Y_2, \dots$ be a branching process generated by a random variable X with mean μ.

(a) Suppose that X has distribution $\mathbb{P}(X = 0) = \frac{1}{4}$, $\mathbb{P}(X = 1) = \frac{1}{4}$ and $\mathbb{P}(X = 2) = \frac{1}{2}$.

(i) State the theorem which allows one to compute $\mathbb{E}(Y_n)$ in terms of the mean value μ of X. Hence compute $\mathbb{E}(Y_3)$. [**3**]

(ii) Explain how one can find the probability of extinction of a branching process and compute this probability. [**5**]

(b) (i) State Markov's inequality. [**3**]

(ii) Use Markov's inequality to prove that if $\mu < 1$ then the probability of extinction of the branching process is 1. [**7**]

Question 4. [**16 marks**] Let $N(t)$ be a Poisson process with intensity $\lambda > 0$ describing the number of customers arriving at a service station during time t.

(a) Give the definition of a Poisson process with intensity $\lambda > 0$. [**5**]

(b) Find the probability that there will be 3 arrivals between times 0 and 2 and no arrivals between times 1 and 3. [**4**]

(c) Let T_2 be the time of the second arrival of a Poisson process. Prove that the probability density function of T_2 is given by

$$f_{T_2}(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & \text{if } x > 0, \\ 0 & \text{if } x \leqslant 0. \end{cases}$$ [**7**]

Question 5. [**12 marks**]

(a) State (do not prove) the Law of Large Numbers. [**4**]

(b) Suppose that you roll a fair die repeatedly. Let S_n be the number of 5's or 6's I see. Prove that

$$\lim_{n\to\infty} \mathbb{P}(0.3n < S_n < 0.4n) = 1$$ [**8**]

End of Paper.