Queen Mary
University of London

MTH5121: Probability Models

Duration: 2 hours

Date and time: 7 June 2016, 14:30–16:30

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You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Examiner(s): I. Goldsheid

Question 1. Two real valued random variables X and Y are jointly continuous with probability density function

$$f_{X,Y}(x,y) = \begin{cases} e^{-y} & \text{if } 0 < x < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

(a) State with reason whether X and Y are independent random variables. **[3]**

(b) Show that the marginal probability density function for Y is
$$f_Y(y) = \begin{cases} ye^{-y} & \text{if } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$ **[8]**

(c) Find the conditional probability density function $f_{X|Y=y}(x)$. **[5]**

(d) Use the conditional probability density function $f_{X|Y=y}(x)$ to compute $\mathbb{E}(X|Y=y)$. Hence express $\mathbb{E}(X|Y)$ in terms of Y. **[3]**

Question 2. Consider a branching process $Y_0, Y_1, Y_2 \dots$ starting with one ancestor, that is $Y_0 = 1$. Suppose that the generating random variable X of this process has the probability mass function $P(X = k) = \binom{m}{k} p^k q^{m-k}$ for $k = 0, 1, 2, \dots, m$, where $0 < p < 1$ and $p + q = 1$.

(a) Find the probability generating function $G(t)$ of the random variable X and use it to compute $\mathbb{E}(X)$. **[5]**

(b) For a given m, find the values of p for which the probability of eventual extinction of this process is 1. **[3]**

(c) Let θ_n be the probability of extinction of this process by time n. State, in terms of $G(t)$, the relation between θ_n and θ_{n-1}. **[3]**

(d) Use the relation between θ_n and θ_{n-1} stated in (c) to express θ_2 in terms of p and q. (You are reminded that $\theta_0 = 0$.) **[4]**

(e) Suppose now that $m = 2$ and $p = \frac{2}{3}$.

(i) Find the probability of eventual extinction for this process. **[6]**

(ii) Suppose that $m = 2$, $p = \frac{2}{3}$ and in addition suppose now that $Y_0 = 3$. What is the probability of eventual extinction in this case? **[3]**

Question 3. Fred has £30 to gamble with at roulette, at a casino. He manages to find a roulette wheel without a 0 (so it has 36 numbers of which 18 are red and 18 are black.) Each time he plays, if black comes up he loses his stake, and if red comes up he wins his stake. He decides to play until he either loses all his money or triples his money to £90.

(a) What is the probability that he triples his money in each of the following two cases?

   (i) The stake is £1 per game. [**3**]

   (ii) The stake is £10 per game. [**3**]

(b) What is the expected length of the game in each of the following two cases?

   (i) The stake is £1 per game. [**4**]

   (ii) The stake is £10 per game. [**4**]

Hint. You are supposed to state and then use formulae for the corresponding probabilities and expectations. You are not required to derive them.

Question 4. Let $N(t)$ be a Poisson Process of rate $\lambda > 0$.

(a) State the definition of the Poisson process of rate $\lambda > 0$. [**5**]

(b) What is the probability that there will be no arrivals between time 3 and time 4, and exactly 3 arrivals between time 3 and time 5? [**5**]

(c) State the theorem describing the joint distribution of inter-arrival times $W_1,\ W_2, ...,\ W_n$ and write down the joint probability density function $f_{W_1,W_2}(x, y)$. [**6**]

Question 5. Let $X_1, X_2 \ldots$ be independent random variables which may not be identically distributed. Suppose that $\mathbb{E}(X_k) = \mu$ for all k and that $\mathrm{Var}(X_k) \leqslant \sigma^2$ for all k. Denote

$$S_n = X_1 + X_2 + ... + X_n \quad \text{and} \quad Y_n = \frac{1}{n} S_n.$$

(a) Prove that $\mathrm{Var}(S_n) \leqslant n\sigma^2$. **[4]**

(b) Prove the following Law of Large Numbers for Y_n: for any $\varepsilon > 0$

$$\mathbb{P}(|Y_n - \mu| \leqslant \varepsilon) \to 1 \text{ as } n \to \infty.$$

Hint. You may use Chebyshev's inequality without proving it. **[9]**

Question 6. (a) State the central limit theorem. **[5]**

(b) 100 devices are used according to the following rule. At each particular moment of time exactly one device is in service and it is replaced by another one as soon as it stops working. The lifetime X_i of the i^{th} device is an exponential random variable with mean of 100 hours. The random variables $X_1, X_2, ..., X_{100}$ are independent.

Use the central limit theorem to find (approximately) the probability that the total working time $S_n = \sum_{i=1}^{n} X_i$ of $n = 100$ devices will be no larger than 9500 hours. Express the answer in the form of an integral of a standard normal probability density function.

Hint. You are reminded that an exponential random variable with mean a has variance a^2. You may use this fact without proving it.

[9]

End of Paper.