Queen Mary University of London

B. Sc. Examination by course unit 2015

MTH5121: Probability Models

Duration: 2 hours

Date and time: 26th May 2015, 10:00–12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Exam papers must not be removed from the examination room.

Examiner(s): I. Goldsheid

Question 1.

(a) Give the definition of a branching process. **[5]**

(b) State the Central Limit Theorem. **[5]**

(c) State (do not prove) Chebyshev's inequality. **[5]**

Question 2. Suppose that X and Y are two random variables.

(a) Define what it means for two random variables X and Y to be independent. **[3]**

(b) State the necessary and sufficient condition for independence of two jointly continuous random variables in terms of their probability density functions. **[3]**

(c) Suppose that the joint density function $f_{X,Y}$ is given by

$$f_{X,Y}(x,y) = \begin{cases} c(x+y) & \text{if } 0 < x, y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Find the constant c. **[5]**

(ii) Find the probability $\mathbb{P}\{X \in [0,\ 0.5] \text{ and } Y \in [0.5,\ 1]\}$. **[7]**

(iii) Find the marginal density functions $f_X(x)$ and $f_Y(y)$. **[5]**

(iv) Are X and Y independent? Justify your answer. **[4]**

(v) Find the conditional density function $f_{X|Y=y}(x)$ and compute $\mathbb{E}(X|Y)$. **[6]**

Question 3. Suppose that a random walk on a line is starting from n, $0 \leqslant n \leqslant N$. The probability of a jump to the right is p and the probability of a jump to the left is $q = 1 - p$. The walk stops once it reaches 0 or N. Let E_n be the expected duration of the walk.

(a) Write down the equations for E_n, where $0 \leqslant n \leqslant N$. **[3]**

(b) Suppose now that $p = q = 1/2$. Write down the solution to the equations from question (a) (no proof is required). **[2]**

(c) Prove the following statement: Suppose that $p = q = 1/2$ and a random walk starts from position 1. Then the expected time until it reaches zero is infinite. **[8]**

Question 4. Let $Y_0, Y_1, Y_2 \ldots$ be a branching process generated by a random variable X with mean μ.

(a) Suppose that X has distribution $\mathbb{P}(X = 0) = \frac{1}{4}$, $\mathbb{P}(X = 1) = \frac{1}{4}$ and $\mathbb{P}(X = 2) = \frac{1}{2}$.

(i) State the theorem which allows one to compute $\mathbb{E}(Y_n)$ in terms of the mean value of X. Hence compute $\mathbb{E}(Y_3)$. **[3]**

(ii) Explain how one can find the probability of extinction of a branching process and compute this probability. **[5]**

(b) Suppose now that X has distribution $\mathbb{P}(X = 0) = \frac{1}{4}$, $\mathbb{P}(X = 1) = \frac{1}{2}$, and $\mathbb{P}(X = 2) = \frac{1}{4}$.

(i) Find $\mathbb{E}(X)$ and $\mathbb{E}(Y_n)$ for all n. **[2]**

(ii) What is the probability of extinction of the branching process in this case? Justify your answer. **[2]**
Hint: no further calculations are necessary in order to answer this question.

(iii) State Markov's inequality. **[3]**

(iv) Prove that $\mathbb{P}(Y_{1000} \geqslant 1000) \leqslant \frac{1}{1000}$. **[3]**

Question 5. Let $N(t)$ be a Poisson process describing the number of customers arriving at a service station during time t.

(a) Give the definition of the Poisson process $N(t)$ with rate $\lambda > 0$. **[5]**

(b) Find the probability that there will be no arrivals during the first 2 units of time and exactly 2 arrivals during the next 1 unit of time. **[5]**

Question 6. State and prove the Law of Large Numbers. You may assume the properties of the variance of a sum of independent random variables but you have to explain what is the property you use. **[11]**

End of Paper.