Queen Mary
University of London

MTH5117: Mathematical Writing

Duration: 2 hours

Date and time: 26th May 2016, 10:00–12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): I. Tomašić

Marks are deducted for incorrect grammar/spelling. In a question, or part of a question, the notation [$\not\in$, n] indicates that the answer should not contain any mathematical symbols whatsoever, apart from numerals. The integer n—when present—prescribes the **approximate** length (in words). In the absence of this notation, mathematical symbols may be used freely.

Question 1 (25 marks). For each of the following mathematical objects provide two levels of description: (i) a coarse description, which only identifies the class to which an object belongs (set, function, etc.) [$\not\in$]; and (ii) a finer description, which describes the object in question as accurately as possible [$\not\in$].

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - x - 1$. **[5]**

(b) $x^3 - x - 1 = 0$. **[5]**

(c) $\{ x \in \mathbb{R} : x^3 - x - 1 = 0 \}$. **[5]**

(d) $\sin^2(x) + \cos^2(x) = 1$. **[5]**

(e) $\sin([-\pi/2, \pi/2])$. **[5]**

Question 2 (25 marks).

(a) Express each of the following statements with symbols, **using at least one quantifier**.

(i) The function $f : \mathbb{R} \to \mathbb{R}$ is not injective. **[4]**

(ii) The function $f : \mathbb{R} \to \mathbb{R}$ assumes all integer values. **[4]**

(iii) The function $\sin : \mathbb{R} \to \mathbb{R}$ is not bounded. **[4]**

(iv) There is no smallest natural number. **[4]**

(v) Every natural number can be written as a sum of four squares of integers. **[4]**

(b) For each statement above, state whether (i) it is definitely true, (ii) it is definitely false, or (iii) there is not enough information to determine whether it is true or false. [You may abbreviate these cases as TRUE, FALSE and UNKNOWN.] **[5]**

Question 3 (16 marks). Consider the implication:

For every real function f, if f is periodic, then f^2 is periodic.

(a) Write its (i) contrapositive, (ii) converse, (iii) negation. **[8]**

(b) For each of (i), (ii), (iii) above, decide whether it is true or false. Justify your claims, providing full proofs or counterexamples, as appropriate. **[8]**

Question 4 (18 marks).

(a) Explain the Principle of Mathematical Induction. **[4]**

(b) The following proof by mathematical induction has at least one fault. State clearly what the faults are. **[6]**

PROPOSITION. For every natural number n,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

PROOF. Assume it is true for n, i.e.,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \tag{1}$$

We need to prove it for $n = n + 1$, i.e.,

$$1 + 2 + \cdots + n + (n+1) = \frac{(n+1)(n+2)}{2}. \tag{2}$$

Subtract equation (1) from (2), and get

$$n+1 = \frac{(n+1)(n+2)}{2} - \frac{n(n+1)}{2} = \frac{n^2 + 3n + 2 - n^2 - n}{2} = \frac{2n+2}{2} = n+1,$$

which is true, so we are done. □

(c) Provide a fully correct proof by mathematical induction of the Proposition from part (b). **[8]**

Question 5 (16 marks). Read the text displayed on the next two pages, and then write a report on it, comprising

- a short title [$\notin$]; **[2]**
- two/three concise key points [$\notin$]; **[4]**
- a summary of the document [$\notin$, 150]. **[10]**

End of Paper—An appendix of 2 pages follows.

This page and the next contain material for Question 5.

Definition. The **greatest common divisor**, or **g.c.d.**, of two positive integers m and n is the largest positive integer which divides both. We write it as $\gcd(m, n)$.

Thus, $\gcd(12, 18) = 6$.

We can extend the notion of greatest common divisor to the case where one of the integers is equal to 0. Since any positive integer divides 0, we see that $\gcd(m, 0) = m$ if $m \neq 0$. If both m and n are zero, then $\gcd(0, 0)$ is undefined (according to our definition above), so we adopt the convention that $\gcd(0, 0) = 0$.

Euclid gave a rule for finding the greatest common divisor of two natural numbers, based on the division algorithm.

Theorem. Let m and n be natural numbers. Then

$$\gcd(m, n) = \begin{cases} m & \text{if } n = 0, \\ \gcd(n, r) & \text{if } m = nq + r \text{ with } 0 \leq r < n. \end{cases}$$

Proof. The first statement is true by definition. For the second, suppose that

$m = nq + r$. Then also $r = m - nq$. So any integer which divides m and n also divides n and r; and **vice versa**; so the greatest common divisor of m and n is equal to the greatest common divisor of n and r. □

This is all very well, but the second line simply replaces the calculation of one g.c.d. by another; why does this help? Notice that $r < n$. This means that, if we apply Euclid's Theorem repeatedly, the second number of the pair of numbers whose g.c.d. we are finding gets smaller at each step. This cannot go on indefinitely; sooner or later, it becomes zero, and the first line applies. So Euclid's Theorem gives us a constructive method to calculate the g.c.d. of two natural numbers. We refer to it as Euclid's Algorithm. (An 'algorithm' is just a constructive method, like a recipe or a set of directions, for achieving some result.) Here it is more formally:

> To find $\gcd(m, n)$:
>
> Put $a_0 = m$ and $a_1 = n$.
>
> As long as the last number a_k found is non-zero, put a_{k+1} equal to the remainder when a_{k-1} is divided by a_k.
>
> When the last number a_k is zero, then the g.c.d. is a_{k-1}.

An example should make it clear.

Example. Find gcd(198, 78).

$$\begin{aligned} a_0 &= 198, a_1 = 78. \\ 198 &= 2 \cdot 78 + 42, \quad \text{so } a_2 = 42. \\ 78 &= 1 \cdot 42 + 36, \quad \text{so } a_3 = 36. \\ 42 &= 1 \cdot 36 + 6, \quad \text{so } a_4 = 6. \\ 36 &= 6 \cdot 6 + 0, \quad \text{so } a_5 = 0. \end{aligned}$$

So $\gcd(198, 78) = 6$.

Euclid's Algorithm actually does more than this. It expresses the greatest common divisor of m and n in terms of the original numbers.

Theorem. For any two natural numbers m and n, there exist integers x and y such that $\gcd(m, n) = xm + yn$.

I will not give a proof here, but here is an example to show how it works. Refer to the preceding example showing that $\gcd(198, 78) = 6$.

Example.

$$\begin{aligned} 6 &= 42 - 36 \\ &= 42 - (78 - 42) = 2 \cdot 42 - 78 \\ &= 2(198 - 2 \cdot 78) - 78 = 2 \cdot 198 - 5 \cdot 78, \end{aligned}$$

so $x = 2$, $y = -5$.

End of Appendix.