Queen Mary
University of London

B.Sc. EXAMINATION BY COURSE UNIT 2012

MTH5117 Mathematical Writing

Duration: 2 hours
Date and time: 30 May 2014, at 10.00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions; marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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EXAMINER: FRANCO VIVALDI

 TURN OVER

Marks are deducted for incorrect grammar/spelling. In a question, or part of a question, the notation [$\not{\in}$, n] indicates that the answer should not contain any mathematical symbols whatsoever, apart from numerals. The integer n —when present— prescribes the *approximate* length (in words). In the absence of this notation, mathematical symbols may be used freely.

Question 1. [Marks: $(5,5,5,5,5),(4,5,5,5),(6)$]

(a) For each of the following mathematical objects, provide two levels of description: 1) a coarse description, which only identifies the class to which the object belongs (set, function, etc.); 2) a finer description, which characterises the object in question as accurately as possible. [$\not\in$]

i) $\{(x,y)\in\mathbb{R}^2 \,:\, x^2=y^2\}$

ii) $\sum_{n\geqslant 0} x^{n^2}$

iii) $(x\in A)\wedge(x\notin B)$

iv) $((a_1),(a_1,a_2),(a_1,a_2,a_3),\ldots)$

v) $\mathbb{Z}\cap f^{-1}(\mathbb{Z})$.

(b) Express each of the following statements with symbols, **using at least one quantifier**.

i) *The real functions f and g are distinct.*

ii) *The sequence (a_k) has precisely one zero term.*

iii) *The real function f is not bounded.*

iv) *Sufficiently close to the origin, the set X has no rational points.*

(c) Consider Goldbach's conjecture:

> *Every even integer greater than 2 can be written as the sum of two primes.*

Write the contrapositive, the converse, and the negation of this statement. [$\not\in$]

Question 2. [Marks: 8,8] Each of the following definitions has faults. *i*) Explain what they are; *ii*) write out an appropriate revision.

(*a*) Let X, Y be sets and let $f, g : X \mapsto Y$ be functions. We define the function f/g as follows:

$$f/g : X \mapsto Y \qquad x \mapsto \frac{f(x)}{g(x)}.$$

(*b*) Let l be a line in the Cartesian plane, let $P, Q \in \mathbb{R}^2$ be the points of intersection of l with the coordinate axes, and let $F(l) : \mathbb{R}^2 \to \mathbb{R}$ be the function that gives the length of the segment joining P and Q.

Question 3. [Marks: 8,8] Explain the following concepts as clearly as you can, in approximately half a page. You may combine words and symbols, and use any material that will assist the reader (examples, theorems, etc.).

(*a*) **Predicate**.

(*b*) **Infinite descent**.

Question 4. [Marks: 2,4,12]

Read the text displayed on the next two pages. Then write a report on it, comprising

i) a short title [$\notin$];

ii) two concise key points [$\notin$];

iii) a summary of the document [$\notin$, 150].

End of paper. **An appendix of 2 pages follows.**

THIS PAGE AND THE NEXT PAGE CONTAIN MATERIAL FOR QUESTION 4.

We wish to characterise the relative rate of growth of functions, as their argument gets large. Let f and g be real functions. We write $f(x) \ll g(x)$ to mean that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$$

and we say that g *dominates* f (or that g has a higher *growth rate* than f) as x goes to infinity. This definition requires that $g(x) \neq 0$ for all sufficiently large x; in what follows we shall assume that this is the case.

Let us begin by comparing powers. For any real numbers a, b with $b > 0$, we have $x^a \ll x^{a+b}$. Indeed:

$$\lim_{x\to\infty} \frac{x^a}{x^{a+b}} = \lim_{x\to\infty} \frac{1}{x^b} = 0.$$

Applying L'Hôpital's Rule, we verify that

$$\lim_{x\to\infty} \frac{\log x}{x^a} = \lim_{x\to\infty} \frac{1}{ax^a} = 0, \qquad a > 0$$

which shows that if a is positive, then $\log(x) \ll x^a$. Likewise, any exponential functions (such as 2^x) grows more rapidly than any power (such as x^5). Indeed, let $a, b \in \mathbb{R}$ with $b > 0$. Repeated applications of Hôpital's Rule give

$$\lim_{x\to\infty} \frac{x^a}{b^x} = \lim_{x\to\infty} \frac{ax^{a-1}}{b^x \log b} = \lim_{x\to\infty} \frac{a(a-1)x^{a-2}}{b^x (\log b)^2} = \cdots$$

Since the exponent at numerator will eventually be zero or negative, we conclude that $\lim_{x\to\infty} x^a/b^x = 0$, that is, $x^a \ll b^x$.

Finally, if $g(x) \to \infty$, then the following holds:

$$f(x) \ll g(x) \quad \Rightarrow \quad e^{f(x)} \ll e^{g(x)}. \tag{1}$$

To see this, we write

$$\frac{e^{f(x)}}{e^{g(x)}} = e^{f(x)-g(x)}$$

and we must prove that $f(x) - g(x) \to -\infty$.

Assume that $g(x)$ dominates $f(x)$, and that $g(x)$ tends to infinity. Then, choosing ϵ such that $0 < \epsilon < 1$, for all sufficiently large x we have $f(x) \leq$

$|f(x)| < \epsilon|g(x)| = \epsilon g(x)$ (g is eventually positive). Thus $f(x) - g(x) < g(x)(\epsilon - 1)$ and since $g(x) \to \infty$, we have $f(x) - g(x) \to -\infty$, as desired.

Note that the converse of implication (1) is false: we have $e^x \ll e^{2x}$, but $x \not\ll 2x$.

The results just established, together with the transitivity of the $\ll$ relation (if $f(x) \ll g(x)$ and $g(x) \ll h(x)$, then $f(x) \ll h(x)$), allow us to order functions according to their growth rate. Let a, b be real numbers with $0 < a < 1 < b$. We find:

$$1 \ll \log(x) \ll \log(x)^b \ll x^a \ll x^b \ll x^{\log(x)} \ll b^x \ll x^x. \tag{2}$$

Let us apply the $\ll$ relation to the evaluation of limits. Consider the limit

$$C = \lim_{x\to\infty} \frac{f(x) + f_1(x) + \cdots + f_n(x)}{g(x) + g_1(x) + \cdots + g_m(x)}$$

where we assume that $\lim_{x\to\infty} f(x)/g(x) = c$, for some c, and that the functions $f_1, \ldots, f_n$ and $g_1, \ldots, g_m$ satisfy the conditions $f_k(x) \ll f(x)$ and $g_k(x) \ll g(x)$ for all k. Collecting $f(x)$ at numerator and $g(x)$ at denominator, we obtain:

$$C = \lim_{x\to\infty} \frac{f(x)}{g(x)} \left(\frac{1 + \frac{f_1(x)}{f(x)} + \cdots + \frac{f_n(x)}{f(x)}}{1 + \frac{g_1(x)}{g(x)} + \cdots + \frac{g_m(x)}{g(x)}} \right) = \lim_{x\to\infty} \frac{f(x)}{g(x)} \cdot \frac{1}{1} = c.$$

We see that the limit is determined solely by the dominant terms. For example, using (2) and the relation $\sin(x) \ll \log(x)$ we have:

$$\lim_{x\to\infty} \frac{x(\log x)^2 - x\sin(x) + \sqrt[4]{x^3}}{x\sqrt{x} + \log(x)} = \lim_{x\to\infty} \frac{x(\log x)^2}{x\sqrt{x}} = \lim_{x\to\infty} \frac{(\log x)^2}{\sqrt{x}} = 0.$$

End of appendix.