Main Examination Period 2018

MTH5110: Introduction to Numerical Computing

Duration: 2 hours

Name:	
Student ID:	

You should attempt ALL questions. Marks available are shown next to the questions.

Answer each question in the corresponding subsection headed "Answer to Problem #" (where # is the problem number), using the provided Maple input or text regions as appropriate. You may also add your own Maple input or text regions, but you should keep each answer in the corresponding subsection. You must use Maple to perform all calculations. Do not delete any relevant input or output; you will score marks only for what is visible in the document you submit. There may be more than one correct solution to each question; any working solution will be accepted provided it satisfies the requirements of the question.

This exam is open book. You may access any information you want, but must work entirely by yourself. You may not communicate, nor attempt to communicate, with anyone else, nor solicit assistance in any way. Please be aware that details of all internet activity on your computer may be logged. You may do rough work on your own paper, which will not be collected by the invigilators. A mobile phone that causes a disruption in the exam is an assessment offence.

Calculators may be used in this examination.

Examiners: T. Popiel, W. Just

Do not read anything below this line until instructed to do so by an invigilator.

Problem 1 [24 marks]

(a) [9 marks]
Compute a floating-point approximation of $\sqrt{8}$ in 5-digit precision.

Compute also the absolute error and the relative error in this approximation.

Answer to Problem 1(a)

(b) [9 marks]

Consider the sequence (a_n) defined by the recursion

$a_n = 3 \cdot a_{n-1} - 2$, with $a_1 = 4$.

Write a procedure, named **mySequence**, that takes as input a positive integer *n* and returns the value of a_n.

Use your procedure to compute a_{20}.

Answer to Problem 1(b)

(c) [6 marks]

It can be shown that the integrals

$$I_n = \int_0^1 \frac{x^n}{\sqrt{2 \cdot x + 1}} \, dx ,$$

where n is a positive integer, satisfy the recurrence relation

$$I_n = \frac{\sqrt{3}}{2 \cdot n + 1} - \frac{n}{2 \cdot n + 1} \cdot I_{n-1}.$$

Explain why the corresponding iteration for computing I_n is stable.

(Note: you do not need to derive the recurrence relation, and you should not need to perform any numerical calculations in order to answer this question.)

Answer to Problem 1(c)

Problem 2 [25 marks]

(a) [15 marks]
Plot the function

$$f(x) = \exp(x) \cdot x^2 - 0.5$$

over the interval $[-4, 1]$. You may assume that all of the roots of $f(x)$ lie in this interval; suppose that you have been asked to find them using the fixed-point iteration method.

Is the function

$$\Phi(x) = f(x) + x$$

a suitable choice of iteration map? Explain your answer.

(Hint: the answer might be "yes" for some roots and "no" for others. You do **not** need to compute the roots.)

Answer to Problem 2(a)

(b) [10 marks]
Consider the function

$$h(x) = 2 \cdot x - 0.7 \cdot \cos(x - 1) - 3 \ .$$

Write a procedure, named **hInverse**, that takes as input a floating-point number *y* and computes the inverse function $h^{-1}(y)$ using the Newton-Raphson method, with absolute error at most 10.0^{-7} .

Use your procedure to compute an approximation of $h^{-1}(4.5)$.

Answer to Problem 2(b)

Problem 3 [25 marks]

(a) [7 marks]
Write a procedure, named **bisectionSolver**, that computes a root of a function via the bisection method.

Your procedure should take as input a function $f(x)$, the endpoints of an interval $[a, b]$ for which either $f(a) < 0$ and $f(b) > 0$ or $f(a) > 0$ and $f(b) < 0$, and an error threshold ε.

The absolute error in the output should be at most ε.

Answer to Problem 3(a)

(b) [10 marks]
Consider the function

$$g(x) = x^3 + x^2 - 4 \cdot x - 2 \cdot$$

Suppose that we wish to use the bisection method to find (all of) the points of **local maximum or local minimum** of $g(x)$.

Choose a suitable initial interval for each point of local maximum or local minimum, and then use your procedure from part (a) with error threshold $\varepsilon = 10.0^{-8}$ to compute these points.

Answer to Problem 3(b)

(c) [8 marks]
Suppose that we wish to use the bisection method to find a root of a function defined on the interval $[-3.1, 8.7]$.

What is the **smallest** number of bisection steps that must be performed in order to guarantee that the absolute error is at most 10^{-6} ? Explain your answer.

(You should assume that the function is continuous and has exactly one root in the given interval.)

Answer to Problem 3(c)

Problem 4 [26 marks]

(a) [10 marks]
Write a procedure, named **myTrapeze**, that uses the trapezoidal rule to compute an approximation of the definite integral

$$\int_0^1 \frac{8}{1+x^2}\, dx \, .$$

Your procedure should have a single input, the number of sub-intervals n.

Answer to Problem 4(a)

(b) [8 marks]
The integral in part (a) is exactly equal to $2 \cdot \pi$. (You do **not** need to prove this.)

Write a procedure, named **minimumNumberOfNodes**, that takes as input a floating-point number ε and computes the minimum value of *n* such that the procedure from part (a) approximates $2 \cdot \pi$ with absolute error less than ε.

Use your procedure to compute the minimum value of *n* such that the procedure from part (a) approximates $2 \cdot \pi$ with absolute error less than 10.0^{-7} .

Answer to Problem 4(b)

(c) [8 marks]
Without performing any further calculations, explain how you would expect the minimum number of sub-intervals determined in part (b) to change if you were to instead approximate the integral using a Riemann sum.

What about if you were to use Simpson's rule?

Answer to Problem 4(c)