Queen Mary
University of London

Main Examination period 2021 – May/June – Semester B
Online Alternative Assessments

MTH5105: Differential and Integral Analysis

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

You have **24 hours** to complete and submit this assessment. When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are expected to spend about 2 hours to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: Huy T. Nguyen, Sasha Sodin

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Question 1 [25 marks].

(a) Let $q : (a, b) \to \mathbb{R}$ be a real valued function. State the definition for q to be **differentiable** at a point $z \in (a, b)$. Give a geometric explanation of what it means to differentiable. **[5]**

(b) Consider the following function, $g : \mathbb{R} \to \mathbb{R}$, given by

$$g(x) = \sqrt{|x|}.$$

Using the definition of the derivative, compute the derivative of g, wherever it exists. Where it does not exist, explain why. **[5]**

(c) Consider the following function, $h : \mathbb{R} \to \mathbb{R}$,

$$h(x) = \begin{cases} 0, & x \leq 0, \\ e^{-1/x^2}, & x > 0. \end{cases}$$

Is h is differentiable on $\mathbb{R}$? If yes, prove it. If not, show that it is not. **[8]**

(d) Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. Prove that f is continuous on (a, b). If f is differentiable, is it true that f' is continuous? If yes, prove the statement, if not, then give a counterexample (and show that it is a counterexample). **[7]**

Question 2 [25 marks].

(a) Prove that $f(x) = \frac{1}{x}$ is uniformly continuous on $[a, 1]$ for any $0 < a < 1$. **[5]**

(b) Let $\{g_n\}_{n=1}^{\infty}$ where $g_n : I \to \mathbb{R}$ be a sequence of functions from an interval I. State the definition for $\{g_n\}_{n=1}^{\infty}$ to be a **uniformly convergent sequence of functions**. Explain how uniformly convergent sequences differ from pointwise convergent sequences. **[5]**

(c) Let $f_n(x) = x^n, \quad x \in [0, 1]$.

(i) Let $f(x) = \lim_{n\to\infty} f_n(x)$. Compute $f(x)$. **[5]**

(ii) Does f_n converge to f uniformly on $[0, a]$, for $a < 1$? Justify your answer. **[5]**

(iii) Does f_n converge to f uniformly on $[0, 1]$? Justify your answer. **[5]**

Question 3 [25 marks].

(a) Let $f : (a, b) \to \mathbb{R}$ be a function that is infinitely differentiable at $x = 0$. State the formula for the Taylor series for f about 0. Is the Taylor series always equal to the function? (If not provide a counterexample, if yes, prove this statement). **[5]**

(b) Let $f(x) = \tan(x)$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Show that f is invertible. Let $g(y) = \arctan(y)$ be the inverse of f and compute the derivative of $g(y)$ in terms of y. **[5]**

(c) Let $h : \mathbb{R} \to \mathbb{R}$ be the function given by

$$h(x) = \frac{1}{1 + x^2}.$$

(i) Using any correct method, find the Taylor series of h about $x = 0$ and find its radius of convergence and interval of convergence. What happens to the Taylor series outside the radius of convergence? What can you say about the function h? **[7]**

(ii) Compute the derivative of h. **[2]**

(iii) Hence, using (ii) or otherwise, find the Taylor series, radius of convergence and interval of convergence for

$$g(x) = \frac{2x}{1 + x^2}$$

about $x = 0$. **[6]**

Question 4 [25 marks].

(a) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. State the definition for the **Upper and Lower integral** of f. Give a criterion for when the Upper integral and Lower integral are equal. **[5]**

(b) Let $f_n(x) = \frac{1}{n}x^n, x \in [0, 1]$. Compute $\lim_{n\to\infty} f_n(x)$ and use this limit to compute

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx.$$ **[5]**

(c) State the **Mean Value Theorem for Integrals**. Explain why it is called the Mean Value Theorem. **[5]**

(d) Let $f : [a, b] \to \mathbb{R}$ denote a bounded function and let F, G be antiderivatives of f. What is the relationship between F and G? Prove any claim that you make. **[5]**

(e) Let h be a continuous function on $\mathbb{R}$ and define

$$H(x) = \int_{x-1}^{x^2} h(t)dt, \quad x \in \mathbb{R}.$$

Show that H is differentiable on $\mathbb{R}$ and compute its derivative H'. **[5]**

End of Paper.