Queen Mary
University of London

B. Sc. Examination by course unit 2015

MTH5105: Differential and Integral Analysis

Duration: 2 hours

Date and time: 28th April 2015, 14:30–16:30

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You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Examiner(s): T. W. Müller

Question 1 (25 marks). (a) Let $f : \mathcal{D} \to \mathbb{R}$ be a real function, and let $a \in \mathcal{D}$ be an inner point of the domain $\mathcal{D} \subseteq \mathbb{R}$. When is f called *differentiable at* a? What is the *derivative of* f *at* a? [**3**]

(b) Straight from the definition of Part (a), show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = -3x^2 + 2$ is differentiable everywhere with derivative $f'(x) = -6x$. [**6**]

(c) State and prove the *product rule for differentiation*. [**6**]

(d) Suppose that the functions $f, g : \mathcal{D} \to \mathbb{R}$ are n-times differentiable on their common domain $\mathcal{D} \subseteq \mathbb{R}$, where n is some positive integer. Show by induction that

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x), \quad x \in \mathcal{D}$$

holds for all $n \geq 1$. You may use without proof the fact that the binomial coefficients satisfy

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}, \quad 1 \leq k \leq n-1$$

for all positive integers n. [**10**]

Question 2 (25 marks). (a) Show: if a function $f : \mathcal{D} \to \mathbb{R}$ is differentiable at a point $a \in \mathcal{D}$, then f is continuous at a. **[3]**

(b) For each positive integer n, exhibit a function $f : \mathbb{R} \to \mathbb{R}$, such that f is n-times differentiable on $\mathbb{R}$, but not $(n+1)$-times. Please justify your answer. **[10]**

(c) Using the differential calculus, prove that

$$\sin(x) > x - \frac{x^3}{6}, \quad x > 0.$$

You may use here without proof the trigonometric formula

$$1 - \cos(x) = 2\sin^2\left(\frac{x}{2}\right),$$

which follows from the addition theorem for cosine, as well as the fact that $\sin(x) < x$ for $x > 0$. **[6]**

(d) (i) State the *mean value theorem of differentiation*. **[2]**

(ii) Show that a function $f : [a, b] \to \mathbb{R}$ satisfying the hypotheses of the mean value theorem, whose derivative is zero on (a, b), is a constant. **[4]**

Question 3 (25 marks). (a) (i) Define what is meant by a *primitive* of a function $f : (a, b) \to \mathbb{R}$. **[2]**

(ii) Show: if a function $f : (a, b) \to \mathbb{R}$ has a primitive $F(x)$, then

$$\{F(x) + c : c \in \mathbb{R}\}$$

is the set of *all* primitives of f. **[6]**

(b) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Define the lower and upper integrals of f, explaining briefly (without proofs) why these quantities are well defined. Please define the terms used in you explanation. When is f as above called *Riemann integrable*, and what is the corresponding *Riemann integral* $\int_a^b f(x)\,dx$? **[10]**

(c) Show that the function $f : [a, b] \to \mathbb{R}$ given by

$$f(x) = \begin{Bmatrix} 0, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q} \end{Bmatrix}, \quad a \le x \le b$$

is *not Riemann integrable*. Here, it is assumed that $b > a$. **[7]**

Question 4 (25 marks). (a) State the *Fundamental Theorem of Calculus*, and use it to compute the integrals

$$\int_1^2 x^n \, dx$$

for all integers $n \geq -1$. [**7**]

(b) State and prove the formula for *partial integration*. [**8**]

(c) Using partial integration, obtain a recurrence relation for the integrals

$$\int \sin^n(x) \, dx$$

with $n \geq 1$. [**5**]

(d) Compute the indefinite integral

$$\int \frac{dx}{x^2 - a^2}$$

for constants $a \neq 0$. [**5**]

End of Paper.