Queen Mary
University of London

Main Examination period 2023 – January – Semester A

MTH5104: Convergence and Continuity

Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

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Question 1 [25 marks].

(a) Prove that the equation

$$x^4 = x + 1$$

does not have any rational solution. [**10**]

(b) Let

$$A = \left\{ \frac{n^2 - 1}{n^3 - 1} : n \in \mathbb{N}, n \neq 1 \right\}.$$

(i) Prove that $A \subseteq \mathbb{R}$ is bounded. [**5**]

(ii) Prove that $\inf A = 0$. Can you replace inf with min? Justify your answer. [**5**]

(iii) Let B be a bounded from above subset of $\mathbb{R}$. Prove that $-B + A$ is bounded from below. [**5**]

Question 2 [25 marks].

(a) Let (a_n) be an increasing sequence of real numbers bounded from above and (b_n) be a decreasing sequence.

(i) Is $(a_n b_n)$ necessarily convergent? Justify your answer. [**5**]

(ii) Prove that if in addition $b_n \geq 1$ for all $n \in \mathbb{N}$ then the sequence $(a_n b_n^{-1})$ is convergent. [**5**]

(b) Let (a_n) be the sequence defined recursively by

$$a_1 = \sqrt{3},$$
$$a_n = \sqrt{2a_{n-1} + 3}, \qquad n \geq 2.$$

(i) Prove that (a_n) is bounded. [**5**]

(ii) Prove that (a_n) is increasing. [**5**]

(iii) Making use of (i) and (ii) prove that the sequence (a_n) is convergent and compute its limit. [**5**]

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Question 3 [25 marks].

(a) Decide whether the following series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, conditionally convergent or divergent. Carefully justify your answer.

(i)
$$a_n = \frac{\sin n}{(n+2)(n+3)} + \frac{2^n + 5^n}{2^n + 9^n};$$
[7]

(ii)
$$a_n = (-1)^n \frac{1}{\sqrt{\sqrt{n} + 4}}.$$
[7]

(b) (i) Find the radius of convergence R of the series
$$\sum_{n=0}^{\infty} (-2)^n n^4 (x-1)^n;$$
[5]

(ii) What can you say for the series (i) when $x = 1 \pm R$? Justify your answer. **[6]**

Question 4 [25 marks].

(a) Prove that the equation
$$x + \ln(\sin(x) + 2) = 1$$
has a solution $x \in \mathbb{R}$. **[15]**

(b) Can you find a continuous function $f : [0,1] \to \mathbb{R}$ such that for all $c > 0$ there exists $x \in [0,1]$ such that
$$|f(x)| > c?$$
Justify your answer. **[10]**

End of Paper.