Queen Mary University of London

Main Examination period 2020 – January – Semester A

MTH5104: Convergence and Continuity

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any unauthorised notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: M. Jerrum, X. Li

You may assume any standard properties of the sine, cosine and exponential functions including the fact that they are continuous.

Question 1 [20 marks].

(a) Suppose A is a non-empty set of real numbers. Define the terms **upper bound** and **supremum** as they apply to A. **[4]**

(b) State the **completeness** axiom for the real numbers $\mathbb{R}$. **[3]**

Let A be a non-empty set of real numbers, and define $B = \{x^2 : x \in A\}$.

(c) Show that B is bounded below. **[3]**

(d) Given an example of a set A such that A is bounded above but B is not. Briefly explain why your example has the desired property. **[4]**

(e) Suppose $\inf(A) \geq 0$ and let $a = \sup(A)$. What is $\sup(B)$? Carefully justify your answer. **[6]**

Question 2 [20 marks]. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers.

(a) Define (using a quantifier expression) what it means for $(x_n)_{n=1}^{\infty}$ to be **bounded**. **[3]**

(b) Define (using a quantifier expression) what it means for a real number $x \in \mathbb{R}$ to be an **accumulation point** of $(x_n)_{n=1}^{\infty}$. **[3]**

(c) Define (using a quantifier expression) what it means for $(x_n)_{n=1}^{\infty}$ to **converge** to 0. **[3]**

(d) Let $(x_n)_{n=1}^{\infty}$ be the sequence defined by $x_n = \sin(1/n)$. Prove directly from the definition that $(x_n)_{n=1}^{\infty}$ converges to 0. You may use the fact that $0 < \sin(x) < x$ for all $x \in (0, 1]$. **[7]**

(e) Let $(x_n)_{n=1}^{\infty}$ be the sequence defined by $x_n = \sin n$. Prove that $(x_n)_{n=1}^{\infty}$ has an accumulation point. You may use any of the standard results of the course, provided it is correctly stated. **[4]**

Question 3 [20 marks].

(a) Define what it means for the **series** $\sum_{k=1}^{\infty} x_k$ **to converge to** S. **[3]**

(b) Show that the series $\sum_{k=1}^{\infty} x_k$ converges in both the following cases:

$$\text{(i)} \quad x_k = \frac{1}{k(k+1)} \quad \text{and} \quad \text{(ii)} \quad x_k = \frac{(-1)^{k+1}}{k}.$$ **[8]**

You do not need to give detailed proofs, as long as the main ideas are made clear.

(c) Consider the statement:

"For every bijection $\phi : \mathbb{N} \to \mathbb{N}$ it is the case that $\sum_{k=1}^{\infty} x_{\phi(k)} = \sum_{k=1}^{\infty} x_k$."

Express the meaning of this statement as precisely as possible using English sentences with few, if any, mathematical symbols. **[4]**

(d) Is the statement in part (c) true of the series (i) and (ii) defined in part (b)? Briefly justify your answer. **[5]**

Question 4 [20 marks]. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, $a \in \mathbb{R}$ be a real number, and $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers.

(a) Define (using a quantifier expression) what it means to say that f is **continuous at** a. **[3]**

(b) Prove: If f is continuous at a and $(x_n)_{n=1}^{\infty}$ converges to a, then $(f(x_n))_{n=1}^{\infty}$ converges to $f(a)$. **[10]**

(c) For $\beta \in \mathbb{R}$, define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ \beta & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Find the limit of $(f(x_n))_{n=1}^{\infty}$ when (i) $x_n = 1/n$ and (ii) $x_n = -1/n$. **[4]**

(d) Deduce that the function f in part (c) is **not** continuous at $a = 0$ for any $\beta \in \mathbb{R}$. **[3]**

Question 5 [20 marks].

(a) State the Intermediate Value Theorem. **[4]**

Now let $f(x) = xe^x + c$, where $c \in \mathbb{R}$.

(b) Briefly explain why the function f is continuous. **[3]**

(c) Prove that, if $c \leq 0$, then the equation $f(x) = 0$ has at least one solution with $x \geq 0$. **[6]**

(d) Prove that, if $0 < c < e^{-1}$, then the equation $f(x) = 0$ has at least two solutions with $x < 0$. You may assume that the sequence $(ne^{-n})_{n=1}^{\infty}$ converges to 0. **[7]**

End of Paper.