Queen Mary University of London

Main Examination period 2022 – May/June – Semester B

MTH5103: Complex Variables

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **2 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: M. Shamis, I. Morris

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Question 1 [20 marks].

(a) Find the real and imaginary parts of $i^{\frac{1}{4}}$, taking the fourth root such that its angle lies between 0 and $\frac{\pi}{2}$. Justify all of your steps. **[10]**

(b) Let z, w be two complex numbers such that $\bar{z}w \neq 1$. Prove that if $|z| = 1$ or $|w| = 1$, then

$$\left|\frac{z-w}{1-\bar{z}w}\right| = 1 \ .$$

[10]

Question 2 [20 marks].

(a) Using the Ratio Test, or otherwise, determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{n^n} z^n.$$

State the test that you are using and justify all of your steps.

[10]

(b) Using the Root Test, or otherwise, determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(n^4-5n^3)^n}{(3n^2-8n^4)^n} (z+i)^n.$$

State the test that you are using and justify all of your steps. **[10]**

Question 3 [20 marks].

(a) Find the coefficients a_n and b_n of the Laurent series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

of the function $f(z) = \frac{1+2z^2}{z^3+z^5}$ valid on the set $\{z \in \mathbb{C} : 0 < |z| < 1\}$, where $z_0 = 0$. Justify all of your steps. **[15]**

(b) Determine the residue of the function $f(z)$ from part (a) at the point $z = 0$. State all the propositions (or theorems) that you are using and justify all of your steps. **[5]**

Question 4 [20 marks].

(a) Express $\mathrm{Log}(2-3i)$ in Cartesian form. Justify all of your steps. [**5**]

(b) Verify that the following function is harmonic

$$u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2 \ .$$

Find a harmonic conjugate v of u, namely a function v such that the function $u(x,y) + iv(x,y)$ is an analytic function. Justify all of your steps. [**15**]

Question 5 [20 marks].

(a) Find all singularities of the function

$$f(z) = \frac{e^{-1/z^3}\sin(z-i)}{(z-i)(z+3)^2},$$

and determine the nature of each of these singularities (e.g. removable singularity, simple pole, double pole, essential singularity). Justify all of your steps. [**10**]

(b) Using the Residue Theorem, or otherwise, compute

$$\int_C \frac{e^{-1/z^3}\sin(z-i)}{(z-i)(z+3)^2}\,dz,$$

where C is the positively oriented circle of radius 6 centered at $z=1$. State all the theorems that you are using and justify all of your steps. [**10**]

End of Paper.