Queen Mary
University of London

MTH5103: Complex Variables

Duration: 2 hours

Date and time: 25th May 2016, 10:00–12:00

Write your solutions in the space provided in this exam paper.

Student number:									
Desk number:									

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You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Examiner(s): S. Beheshti

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Question	Mark	Subpart Breakdown
1		
2		
3		
4		
5		
6		
TOTAL :		

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Question 1.

(a) Describe graphically the set of points z in the complex plane satisfying $\Im\left(z^3\right) \geq 0$. Justify your answer.

[**4**]

Write your solution to Question #1(a) below

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(b) Define what is meant by a Möbius transformation. Determine the unique Möbius transformation which sends $z = -1$ to $w = i$, $z = \infty$ to $w = 1$, and $z = i$ to $w = 1 + i$. Verify that, for this Möbius transformation, the real axis in the z-plane is mapped onto the circle of radius 1 centred at the origin in the w-plane.

[**6**]

Write your solution to Question #1(b) below

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(c) Write down the *Cauchy–Riemann equations* satisfied by the real and imaginary parts u and v of a complex function $f(z)$ and state the conditions under which this f is guaranteed to be complex differentiable at z_0.

[**3**]

(d) Let $f(z) = e^{-iz}$ be a function of a complex variable $z = x + iy$. Use part (c) to show that $f'(z)$ exists for all z.

[**3**]

Write your solution to Question #1(c) and #1(d) below

Additional space for Question 1

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Question 2.

(a) Find the Taylor series expansion of the function $f(z) = \dfrac{z^3}{z+4}$ about $z_0 = 0$ and determine the radius of convergence of the series. [**6**]

Write your solution to Question #2(a) below

(b) Suppose the power series $\sum_{n=0}^{\infty} a_n(z-2)^n$ is known to have radius of convergence $R = 1$. What can be said about the convergence/divergence of $\sum_{n=0}^{\infty} \frac{a_n}{2^n}$? Justify your answer. [**6**]

Write your solution to Question #2(b) below

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(c) Give an example, if possible, of a power series centred at $z_0 = 0$ which converges for all z with $\Im(z) = 1$ but diverges for all other $z \in \mathbb{C}$. If there is no possible example, explain why not. [**4**]

Write your solution to Question #2(c) below

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Additional space for Question 2

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Question 3. Consider the function $f(z) = \dfrac{12}{z(z+4)}$.

(a) Find the coefficients a_n and b_n of the Laurent series

$$\sum_{n=0}^{\infty} a_n (z+4)^n + \sum_{n=1}^{\infty} b_n (z+4)^{-n}$$

of $f(z)$ on a punctured disc centred at $z_0 = -4$ and specify the region on which the series is valid. You should also indicate what is meant by a punctured disc. [**6**]

Write your solution to Question #3(a) below

(b) Using part (a), what type of singularity does $f(z)$ have at the point $z_0 = -4$? [**6**]

Write your solution to Question #3(b) below

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(c) Determine the residue of $f(z)$ at the point $z_0 = -4$. [**6**]

Write your solution to Question #3(c) below

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Additional space for Question 3

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Question 4.

(a) Explain what is meant by an *isolated singularity* of a complex function f. Locate the singularities of $f(z) = z^5 \sin\left(\frac{1}{z}\right)$ and determine the nature of these singularities (e.g., pole of order m, removable singularity or essential singularity).

[**6**]

Write your solution to Question #4(a) below

(b) Prove the following: If $f(z)$ has a zero of order m at $z_0 = 0$, then $g(z) = \frac{1}{f(z^2)}$ has a pole of order $2m$ at $z_0 = 0$.

[**6**]

Write your solution to Question #4(b) below

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(c) Determine the singularities of $f(z) = \dfrac{e^{-i\pi z}}{z^2 - 9}$ and compute the residue of f at each such singularity. [**6**]

Write your solution to Question #4(c) below

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Additional space for Question 4

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Question 5.

(a) Let C describe the unit circle traversed once, anti-clockwise. Using the Estimation Theorem (also called the M-L Inequality), show that

$$\left| \int_C \frac{e^z}{4z^4} \, dz \right| \leq \frac{\pi e}{2} \, .$$

[**7**]

Write your solution to Question #5(a) below

(b) State Cauchy's Theorem. [**5**]

Write your solution to Question #5(b) below

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(c) Consider the closed, anticlockwise-oriented triangle C, comprised of the union of the three line segments joining the points $e^{i\pi/4}$, -2, and $\frac{1}{2} - i$ in the complex plane. Draw the path given and use Cauchy's Theorem to evaluate

$$\int_C \frac{5z^2}{81 - z^4}\, dz.$$

[**6**]

Write your solution to Questions #5(c) below

Additional space for Question 5

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Question 6.

(a) State the Residue Theorem. [**5**]

Write your solution to Question #6(a) below

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(b) Using the Residue Theorem, or otherwise, evaluate

$$\int_C \frac{\cos \frac{i\pi z}{2}}{(z+i)(z-3)^2}\, dz,$$

where C is the positively oriented circle of radius 2 centred at the origin. [**9**]

Write your solution to Question #6(b) below

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Additional space for Question 6

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This page is for additional work and will NOT be marked.

End of Paper.

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