Queen Mary
University of London

B. Sc. Examination by course unit 2014

MTH5103 Complex Variables

Duration: 2 hours

Date and time: 14th May 2014, 10:00 –12:00

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You should attempt all questions. Marks awarded are shown next to the questions.

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Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Examiner(s): O. M. Jenkinson

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Question 1 (a) Find all solutions $z \in \mathbb{C}$ of the equation $z^3 = 8$. [5]

(b) Find all solutions $z \in \mathbb{C}$ of the equation $e^{3z} = 1$. [5]

(c) Suppose $w = f(z) = \frac{z-1}{z-3}$. Show that the image under f of the line $Re(z) = 2$ is the unit circle $\{w \in \mathbb{C} : |w| = 1\}$. [5]

Question 2 (a) Write down the definition of the *radius of convergence* of a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$. [3]

(b) Find the Taylor series expansion $\sum_{n=0}^{\infty} a_n (z-3)^n$ of $f(z) = 1/(1+4z)$ about the point $z_0 = 3$. [5]

(c) Determine the radius of convergence of the Taylor series in (b) above. [2]

(d) If $g(z) = \frac{1}{(z-1)(3-z)}$, find the Laurent series $\sum_{n=0}^{\infty} a_n (z-1)^n + \sum_{n=1}^{\infty} b_n (z-1)^{-n}$ for g about the point $z = 1$. [5]

Question 3 (a) Write down the *Cauchy-Riemann equations* satisfied by the real and imaginary parts u and v of a complex function $f = u + iv$ at any point z_0 where f is complex differentiable. [3]

(b) If u and v satisfy the Cauchy-Riemann equations at z_0, what extra condition on u and v will ensure that f is complex differentiable at z_0 ? [2]

(c) Let $f(z) = y(3x^2 - y^2) + ix(x^2 - 3y^2)$. Show that f is complex differentiable at just one point, and compute its derivative at this point. [5]

(d) If $f : \mathbb{C} \to \mathbb{C}$ is entire, and $Re(f(z)) = Im(f(z))$ for all $z \in \mathbb{C}$, use the Cauchy-Riemann equations to prove that f is a constant function. [5]

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Question 4 (a) What is meant by an *isolated singularity* of a complex function f ? What does it mean to say that such a singularity is a *pole of order* m ? What is meant by the *residue* of f at an isolated singularity ? [6]

(b) Locate the poles of the function

$$f(z) = \frac{z}{(z^2+9)^2} \,.$$

Determine the order of each pole, and its residue. [6]

(c) State Rouché's Theorem (without proof). How many zeros (counted with multiplicity) does the polynomial $p(z) = z^6 - 5z^5 + 10z^4 - 2z^2 + 1$ have in the annulus $\{z \in \mathbb{C} : |z| \geq 1\}$? Justify your answer. [8]

Question 5 (a) Let C be a contour parametrised by a piecewise smooth function $\gamma : [a,b] \to \mathbb{C}$. Define what is meant by the *contour integral* $\int_C f(z)\,dz$ of the complex function f along the contour C. [4]

(b) Evaluate the integral $\int_C f(z)\,dz$ when $f(z) = \bar{z}$ (the complex conjugate of z) and

(i) C is the straight line segment from $+2$ to -2. [4]

(ii) C is the curve from $+2$ to -2 along the upper half of the radius-2 circle centred at 0. [4]

(c) Is it possible that the function $f(z) = \bar{z}$ has an *antiderivative* on $\mathbb{C}$? Find one or else give a reason why such an antiderivative cannot exist. [3]

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Question 6 (a) State Cauchy's Theorem and use this to calculate

$$\int_C \frac{\sin(z)}{z^2 + 2z + 2}\, dz$$

where C is the positively oriented circle centred at the origin with radius 1, being careful to justify your answer. [8]

(b) State the Residue Theorem. [2]

(c) Using the Residue Theorem, or otherwise, evaluate

$$\int_C \left(e^{1/z} + \frac{1}{z-1} \right) dz$$

where C is the positively oriented circle having centre $z = 1$ and radius 2. [10]

End of Paper

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