

## B. Sc. Examination by course unit 2014

## MTH5102 Calculus III

Duration: 2 hours

Date and time: 19 May 2014, 10:00 am

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You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Examiner(s): L. Lacasa

Question 1 [22]

Let C a curve in  $\mathbf{R}^3$  whose parametric equation reads  $\mathbf{r}(t) = (\cos t, \sin t, 2t)$ , and consider the points A, B, C and D whose coordinates are, respectively,  $A = (1, 0, 0), B = (0, 1, \pi), C = (1, 1, 1)$  and  $D = (-1, 0, 2\pi)$ .

- (a) Describe what kind of curve C is, and make a sketch. [4]
- (b) Briefly justify which points from  $\{A, B, C, D\}$  belong to the curve  $\mathcal{C}$ . [4]
- (c) Calculate the arc length of C, from A to D. [6]
- (d) Consider the vector field  $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$ . Calculate the line integral of  $\mathbf{F}$  over  $\mathcal{C}$ , between A and D.

Question 2 [21]

Let U be a suitably differentiable scalar field, and  $\mathbf{F}$  be a suitably differentiable vector field.

- (a) Write down the expressions for  $\nabla U$ ,  $\nabla \cdot \mathbf{F}$  and  $\nabla \times \mathbf{F}$  in Cartesian coordinates. [3]
- (b) For each of the following expressions, state (without proof) whether it is (i) a scalar field (ii) a vector field or (iii) not a valid expression:

$$U(\nabla \times \mathbf{F}) \; ; \quad U(\nabla \cdot \mathbf{F}) \; ; \quad \nabla \times (\nabla U) \; ; \quad \nabla \times (\nabla \cdot \mathbf{F}) \; \; .$$

[4]

(c) Let  $\mathbf{F}: \mathbf{R}^3 \to \mathbf{R}^3$  a vector field. Prove that  $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ . [5]

Let  $\mathbf{F} = (y+z)\mathbf{i} + x\mathbf{j} + (x+2z)\mathbf{k}$ .

- (d) Is **F** solenoidal? Justify your answer. [2]
- (e) Give the definition of a scalar potential, saying what condition is needed for a scalar potential to exist. Show that this condition holds for **F**, and calculate a scalar potential for **F**.

Question 3 [12]

- (a) State (without proof) the Divergence theorem (define the terms used and any required conditions). [4]
- (b) Apply this theorem to calculate the surface integral of the vector field  $\mathbf{F} = y\mathbf{i} + 2y\mathbf{j} 3\mathbf{k}$  over a cylinder such that  $x^2 + y^2 < a^2$ , 0 < z < b, where a, b are positive constants. Explain all the assumptions you make to apply the theorem.

Question 4 [11]

Consider the position vector  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ .

- (a) Express the cartesian coordinates (x, y, z) in terms of spherical coordinates  $(r, \theta, \phi)$ .
- (b) Express the spherical coordinate unit vectors  $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$  in terms of the cartesian unit vectors  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ . [3]
- (c) Consider the vector field  $F = -z\mathbf{k}$  in the surface of a sphere of radius a with vector area element  $d\mathbf{S}$ . Calculate the scalar product  $\mathbf{F} \cdot d\mathbf{S}$ . [6]

Question 5 [16]

Let f(x) be a periodic function of period  $2\pi$  defined in  $(-\pi,\pi)$  by

$$f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi. \end{cases}$$

- (a) State (without proof) the general expression for a Fourier series S(x). [2]
- (b) Show (without calculating the terms) that the Fourier series of f(x) does not have any  $\cos nx$  terms. [3]
- (c) Calculate its Fourier series. [7]
- (d) By suitably evaluating the Fourier series and by virtue of Dirichlet's theorem, calculate the sum  $\sum_{k=0}^{\infty} (-1)^k/(2k+1)$ . [4]

Question 6 [9]

Consider the Laplace equation  $\nabla^2 \phi = 0$  over a rectangle in the XY plane, subject to some boundary conditions.

Briefly explain the method of separation of variables, and justify why this method proposes that a possible solution to the problem takes the form

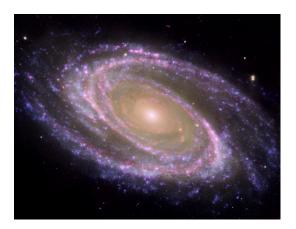
$$\phi(x,y) = (A\cos(kx) + B\sin(kx))(C\cosh(ky) + D\sinh(ky)),$$

where A, B, C, D are constants that depend on the boundary conditions.

Question 7 [9]

The Milky Way (see picture below) is a rotating galaxy. Let us define a function V that describes the instantaneous velocity of a planet. The location of this planet is referenced to the centre of the galaxy, where we put the origin of the coordinate system.

- (a) Justify why V is a vector field. [2]
- (b) Explain how could you describe the average direction and magnitude of rotation of the Milky Way close to its centre. [4]
- (c) Assuming that the Milky Way is approximately a planar galaxy which belongs to the XY plane, calculate the average direction of the rotation close to its centre.



**End of Paper**