Queen Mary
University of London

Main Examination period 2020 – May/June – Semester B
Online Alternative Assessments

MTH5101: Ring Theory

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment, you may use books, notes, and the internet. You may use calculators and computers, but you should show your working for any calculations you do. You must not seek or obtain help from anyone else.

At the start of your work, please **copy out and sign** the following declaration:

> I declare that my submission is entirely my own, and I have not sought or obtained help from anyone else.

All work should be **handwritten**, and should **include your student number**.

You have **24 hours** in which to complete and submit this assessment. When you have finished your work:

- scan your work, convert it to a **single PDF file** and upload this using the upload tool on the QMplus page for the module;
- e-mail a copy to maths@qmul.ac.uk with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are not expected to spend a long time working on this assessment. We expect you to spend about **2 hours** to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the assessment period, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: F. Rincón, S. Majid

Question 1 [24 marks].

(a) Define what is meant by a **ring homomorphism** $\theta : R \to S$ between rings R, S, and define what is meant by the **image** $\mathrm{Im}(\theta)$ and by the **kernel** $\mathrm{Ker}(\theta)$ of θ. **[4]**

(b) Give an example of a map $f : \mathbb{Z} \to \mathbb{Z}$ that is **not** a ring homomorphism. Justify your answer. **[4]**

(c) Consider the map $\phi : \mathbb{Z} \to \mathbb{Z}/12\mathbb{Z}$ defined as $\phi(m) = [4m]_{12}$.

(i) Prove that ϕ is a ring homomorphism. **[6]**

(ii) Describe explicitly the image $\mathrm{Im}(\phi)$ and the kernel $\mathrm{Ker}(\phi)$ of ϕ. How many elements do $\mathrm{Im}(\phi)$ and $\mathrm{Ker}(\phi)$ have? **[6]**

(iii) Use the First Isomorphism Theorem to express your answer for $\mathrm{Im}(\phi)$ in part (ii) as isomorphic to a factor ring of $\mathbb{Z}$. **[4]**

Question 2 [23 marks].

(a) Suppose I is an ideal of a ring R. Explain what the **factor ring** R/I is, by describing its elements and the way that addition and multiplication are defined for them. You do not need to prove anything. **[4]**

(b) Explain why it is required that I is an ideal of R (as opposed to just a subring) in order to construct the factor ring R/I. **[4]**

(c) Consider the ring $R = 3\mathbb{Z}$ and the ideal $I = 12\mathbb{Z}$ of R.

(i) Give a representative for each coset of I in R. **[4]**

(ii) Is R/I a field? Justify your answer. **[5]**

(iii) Use the Second Isomorphism Theorem to list all the subrings of R/I. **[6]**

Question 3 [17 marks].

(a) Define what is meant by a **zero-divisor** and a **unit** in a domain R. Illustrate each definition with an example. **[6]**

(b) Prove that if u is a zero-divisor in a domain R then u is not a unit. **[5]**

(c) Consider the Boolean ring $R = \mathcal{P}(\{a, b, c\})$, with addition being symmetric difference and multiplication being intersection. List all the zero-divisors of R, and all the units of R. [The symmetric difference of two sets A and B is $A \Delta B = (A \cup B) \setminus (A \cap B)$.] **[6]**

Question 4 [20 marks]. For each of the following statements, write down whether it is true or false. Briefly justify your answers.

(a) The ring $4\mathbb{Z}$ has cardinality 4. **[4]**

(b) If X is any non-empty set, the Boolean ring $\mathcal{P}(X)$ is a commutative ring with identity. **[4]**

(c) The ring $\mathbb{Z}/6\mathbb{Z}$ has exactly 4 subrings. **[4]**

(d) Any subring with identity of an integral domain must also be an integral domain. **[4]**

(e) The rings $\mathbb{F}_2[x]/\langle x^2 \rangle$ and $\mathbb{F}_2[x]/\langle x^2 + x \rangle$ are isomorphic (where $\mathbb{F}_2$ denotes the field with 2 elements). **[4]**

Question 5 [16 marks].

(a) Define what is meant by a **maximal ideal** of a ring R, and give an example of a maximal ideal of $\mathbb{Z}$. **[4]**

(b) Is the ideal $\langle x^3 - 2 \rangle$ a maximal ideal of $\mathbb{Q}[x]$? Justify your answer. [You can assume that $\sqrt[3]{2}$ is not a rational number.] **[6]**

(c) Find the inverse of the element $[1 + x]$ in the factor ring $\mathbb{Q}[x]/\langle x^3 - 2 \rangle$. **[6]**

End of Paper.