Queen Mary
University of London

Main Examination period 2017

MTH5100: Algebraic Structures I

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiners: S. Majid

Question 1. [18 marks]

(a) Define what it means for a ring R to be **commutative**. [**2**]

(b) Show that the ring $M_2(\mathbb{F}_2)$ of 2×2 matrices with entries from $\mathbb{F}_2$ is **non**-commutative. Here $\mathbb{F}_2$ is the field of 2 elements. [**4**]

(c) Define what it means for a subset $S \subseteq R$ of a ring R to be a **subring**. [**3**]

(d) Define what it means for a subring $I \subseteq R$ of a ring R to be an **ideal**. [**3**]

(e) Give an example of a subring of $M_2(\mathbb{F}_2)$ which is **not** an ideal. Justify your answer. You may use one of the subring tests from lectures. [**6**]

Question 2. [15 marks]

(a) Define what it means for a ring R to have an **identity**. [**2**]

(b) Give an example of a subring $S \subseteq R$ of a ring R where S has an identity which is **not** an identity for R. Justify your answer regarding the identity. [**6**]

(c) Let $I \subseteq R$ be an ideal of a ring R. Suppose that R has identity 1 and that $1 \in I$. Prove that $I = R$. [**3**]

(d) Let $I \subseteq R$ be an ideal of a **field** R. Prove that $I = \{0\}$ or $I = R$. Hint: show that if I has a nonzero element i then $ii^{-1} \in I$. [**4**]

Question 3. [14 marks]

(a) Let $\theta : \mathbb{Z}_8 \to \mathbb{Z}_4$ be defined by $\theta([i]_8) = [i]_4$ for all $i \in \mathbb{Z}$. You may assume that this is a well-defined ring homomorphism. Find $\mathrm{Ker}(\theta)$ and exhibit the partition of $\mathbb{Z}_8$ into cosets of $\mathrm{Ker}(\theta)$. [**6**]

(b) Let $I \subseteq R$ be an ideal of a ring R. What are the elements of the **factor ring** R/I and how do you add and multiply them to form a ring? You are not asked to prove anything. [**3**]

(c) For θ in part (a), find a nonzero element of $\mathbb{Z}_8/\mathrm{Ker}(\theta)$ which squares to zero. Show that the factor ring here is isomorphic to $\mathbb{Z}_4$. You may use any results from lectures. [**5**]

Question 4. [26 marks]

(a) Give an example in $\mathbb{Z}_8$ of a **zero divisor** and of a **unit** other than the identity. [**4**]

(b) In a commutative ring with identity, state what it means for two elements to be **associates** and for an element to be **irreducible**. [**4**]

(c) State what is meant by each of the terms:

(i) Integral domain; [**4**]

(ii) Unique factorisation domain; [**4**]

(iii) Principal ideal domain; [**4**]

(iv) Euclidean domain. [**4**]

(d) State an example of a unique factorisation domain which is **not** a principal ideal domain. You are not asked to prove anything. [**2**]

Question 5. [13 marks]

(a) Factorise $4 \in \mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} \mid a, b \in \mathbb{Z}\}$ in two different ways to show that $\mathbb{Z}[\sqrt{-3}]$ is **not** a unique factorisation domain. You may assume that its only units are ± 1 and that $2, 1 \pm \sqrt{-3}$ are irreducible. [**4**]

(b) State a Euclidean function d that makes the Gaussian integers $\mathbb{Z}[\sqrt{-1}] = \{a + b\sqrt{-1} \mid a, b \in \mathbb{Z}\}$ into a Euclidean domain. [**2**]

(c) Show that every Euclidean domain is a principal ideal domain. [**5**]

(d) Is every principal ideal domain a unique factorisation domain? You are not asked to justify your answer. [**2**]

Question 6. [14 marks]

(a) State what it means for an ideal $I \subseteq R$ of a ring R to be **maximal**. [**3**]

(b) Show that $f = 1 + x + x^2 \in \mathbb{F}_2[x]$ is irreducible and outline why $\langle f \rangle$ is a maximal ideal of $\mathbb{F}_2[x]$ given that the latter is a principal ideal domain. [**6**]

(c) For f in part (b), prove that $\mathbb{F}_2[x]/\langle f \rangle$ is a finite field of 4 elements. You may use any general results from lectures. [**5**]

End of Paper.