Queen Mary
University of London

B. Sc. Examination by course unit 2015

MTH4110: Mathematical Structures

Duration: 2 hours

Date and time: 19 May 2015, 14:30–16:30

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You should attempt ALL questions. Marks awarded are shown next to the questions.

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Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Examiner(s): O. F. Bandtlow

Question 1 (10 marks).

(a) Let a and b be integers. What does it mean to say that *a divides b*? **[2]**

(b) Suppose that $c \geq 2$ is a natural number which is not prime. Show that there is a natural number $d \geq 2$ such that $d \mid c$ and $d \leq \sqrt{c}$. **[8]**

Question 2 (10 marks).

(a) Let Ω be a set, and let A, B and C be subsets of Ω. Define the following sets:

(i) $A \cup B$, (ii) $A \cap B$, (iii) $A \setminus B$, (iv) $A \triangle B$. **[4]**

(b) Consider the following statements.

(i) If A, B and C are sets, then $A \cup (B \cap C) = (A \cup B) \cap C$.
(ii) If A, B and C are sets with $A \subseteq B$, then $C \setminus B \subseteq C \setminus A$.

For each of them decide whether it is true or false. If it is true prove it without appealing to Venn diagrams; if it is false give a counterexample. **[6]**

Question 3 (10 marks). Let A be a set.

(a) Explain what is meant by the *power set $\mathscr{P}(A)$ of A*. **[2]**

(b) Prove that there is no matching between A and $\mathscr{P}(A)$. **[6]**

(c) Is the set of sets of natural numbers countable? Give reasons for your answer. **[2]**

Question 4 (10 marks).

(a) Let R be a relation on a set A. Explain what is meant by saying that R is

(i) *reflexive*, (ii) *symmetric*, (iii) *transitive*. **[3]**

(b) Give an example of a relation on $\mathbb{N}$ which is reflexive and transitive, but not symmetric. **[2]**

(c) Let A and B be sets, and let $f : A \to B$ be a function. Define a relation R on A by setting $a_1 \, R \, a_2$ if $f(a_1) = f(a_2)$. Show that R is an equivalence relation on A. **[5]**

Question 5 (10 marks). Prove, using induction, that $1 + 2n \leq 3^n$ for every $n \in \mathbb{N}$.

Question 6 (10 marks).

(a) Let a and b be natural numbers. Explain what is meant by the *greatest common divisor of a and b*. **[2]**

(b) Use Euclid's algorithm to find the greatest common divisor of 69 and 78. **[8]**

Question 7 (10 marks).

(a) Explain what is meant by a *rational number* and an *irrational number*. **[2]**

(b) Let x and y be real numbers. Prove that if xy is irrational, then x or y is irrational. **[6]**

(c) Give an example of an irrational number (proof of irrationality not required). **[2]**

Question 8 (10 marks).

(a) Let

$$z = \frac{1}{\sqrt{2}}(1+\mathrm{i}).$$

Determine the modulus and argument of z. Hence, or otherwise, find the real and imaginary part of z^8. **[4]**

(b) Let $F : \mathbb{C} \to \mathbb{C}$ be given by $F(z) = z^8$.

(i) Is F injective?

(ii) Is F surjective?

In each case, give reasons for your answers. **[6]**

Question 9 (10 marks). Let n and m be natural numbers. Consider the following statement.

If n is prime and m is odd, then nm is odd.

(a) Write down the contrapositive. **[2]**

(b) Write down the converse. **[2]**

(c) Is the statement true? Is the contrapositive true? Is the converse true? Give reasons for your answers. **[6]**

Question 10 (10 marks). Find the mistakes in the following proof.

Theorem *For every natural number n, the number* $4^n - 1$ *is divisible by* 3.

Proof For a natural number n, let $P(n)$ be the statement that 3 divides $4^n - 1$.
We start by observing that $P(1)$ is true, since $4^1 - 1 = 3$ which is divisible by 3.
Suppose now that $P(n)$ is true for every $n \in \mathbb{N}$. Thus $4^n - 1 = 3k$ for some natural number k. Then

$$4^{n+1} - 1 = 4 \cdot 4^n - 1 = 4(3k+1) - 1 = 12k + 3 = 3(4k+1)$$

which is divisible by 3. Thus $P(n+1)$ holds.
Thus, since $P(1)$ is true, and $P(n)$ and $P(n+1)$ are true, the statement $P(n)$ is true for all natural numbers n, by the principle of induction. □

End of Paper.