Queen Mary
University of London

B. Sc. Examination by course unit 2015

MTH4104: Introduction to Algebra

Duration: 2 hours

Date and time: 11 May 2015, from 14:30

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You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Examiner(s): A. R. Fink

Question 1.

(a) Give the definition of a *partition* of a set X. **[3]**

(b) Let $\{A_1, A_2, \ldots\}$ be a partition of a set X, and R the relation

$$\{\, (x, y) \in X^2 : \text{ there exists } j \text{ such that } x \in A_j \text{ and } y \in A_j \,\}.$$

Prove that R is an equivalence relation. **[6]**

Question 2.

(a) Prove that $[65]_{186}$ has a multiplicative inverse in the ring $\mathbb{Z}_{186}$. **[6]**

(b) Compute this multiplicative inverse. **[8]**

(c) How many of the elements of $\mathbb{Z}_{186}$ have multiplicative inverses? Justify your answer. **[6]**

Question 3. Let f be the permutation $(1\ 10\ 3\ 9\ 7\ 4)(2)(5\ 11\ 8)(6)$ in S_{11}, which is written in cycle notation.

(a) Write f in two-line notation. **[3]**

(b) Let g be the element

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 8 & 5 & 1 & 6 & 4 & 11 & 9 & 7 & 10 & 3 \end{pmatrix}$$

of S_{11}, written in two-line notation. Determine $(gf)^{-1}$, and write your answer in cycle notation. **[6]**

(c) Write down an element of S_{11} of order 21. **[4]**

Question 4.

(a) State the definition of the complex number $\mathrm{e}^{\mathrm{i}\theta}$, where θ is a real number. **[2]**

(b) Prove that $\mathrm{e}^{\mathrm{i}\theta} \cdot \mathrm{e}^{\mathrm{i}\phi} = \mathrm{e}^{\mathrm{i}(\theta+\phi)}$ for all real numbers θ and ϕ. **[4]**

(c) Prove by mathematical induction, or otherwise, that for all integers $n \geqslant 1$,

$$\cos(1) + \cos(2) + \cdots + \cos(n-1) = \frac{\cos(n) - \cos(n-1)}{2\cos(1) - 2} - \frac{1}{2}.$$ **[9]**

Question 5.

(a) Let R be a ring. Prove that $-(ab) = (-a) \cdot b$ for any elements $a, b \in R$. **[6]**

(b) Let R be a ring, and define the relation $|$ on R so that, if a and b are elements of R, then $a \mid b$ if and only if $b = ra$ for some $r \in R$. Must the relation $|$ be reflexive? symmetric? transitive? Prove your assertions. **[6]**

Question 6. Let S be the subset of $\mathrm{M}_2(\mathbb{C})$ consisting of matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

(a) Prove that S is closed under addition and multiplication. **[4]**

(b) Prove that S satisfies the multiplicative inverse law. You may assume that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the multiplicative identity in S. **[6]**

(c) Prove that S is not a field. **[6]**

Question 7.

(a) Define what it means for a set G with an operation $\circ$ to be a *group*. **[3]**

(b) Give an example of two finite groups which have the same order but are not isomorphic. **[6]**

(c) Let R be a ring with identity. Prove that the set $R^\times$ of units of R, with the operation of multiplication, is a group. **[6]**

End of Paper.