Queen Mary
University of London

B. Sc. Examination by course unit 2014

MTH4104 Introduction to Algebra

Duration: 2 hours

Date and time: 28 April 2014, beginning 14:30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions.
Marks awarded are shown next to the questions.

Calculators must not be used.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): Dr Alex Fink

Question 1 (a) Give definitions of the terms
(i) *relation*; [2]
(ii) *equivalence relation*. [3]

(b) Give an example of an equivalence relation on the set $\{1,2,3\}$ with exactly two equivalence classes. [3]

Question 2 (a) Use the Euclidean algorithm to compute $\gcd(426,330)$. [6]

(b) Find a solution to the equation

$$426k + 330\ell = \gcd(426,330)$$

where k and ℓ are integers. [8]

Question 3 Solve the following system of equations over $\mathbb{Z}_{11}$ for x and y.

$$[4]_{11}\,x + [7]_{11}\,y = [4]_{11}$$
$$[2]_{11}\,x + [6]_{11}\,y = [1]_{11}.$$

Justify your answer. [8]

Question 4 Let f be the following permutation in S_{10}, given in two-line notation.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 7 & 9 & 6 & 8 & 1 & 5 & 10 & 3 & 2 \end{pmatrix}$$

(a) Write f in cycle notation. [3]

(b) Let $g \in S_{10}$ be the element $(1)(2\ 8\ 6\ 7)(3\ 5\ 4\ 9)(10)$, in cycle notation. Determine fg^{-1}, written in cycle notation. [6]

(c) Determine the order of f. [3]

(d) Specify an integer n such that f^n fixes exactly seven elements of the set $\{1,2,\ldots,10\}$. [4]

Question 5 (a) State the definition of the *divisibility relation* $\mid$ on the set of natural numbers. [3]

(b) Prove, using mathematical induction, that

$$12 \mid (7^n - 3^{n+1} + 2)$$

for all natural numbers $n \geq 0$. [9]

Question 6 (a) Let R be a set on which two operations $+$ and $\cdot$ are defined. Define what it means for R to be a *ring*. [4]

(b) Let R be a ring. Prove that, if 0 is the additive identity in R, then $0 \cdot a = 0$ for every element a of R. [4]

(c) Give an example of a ring whose set of elements is finite and in which the commutative law for multiplication does not hold. Justify your answer. [6]

Question 7 (a) Let G be a group. Define what it means to say that a set H is a *subgroup* of G. [3]

(b) Let g and h be elements of a group G. Prove that if $gh = hg$, then $g^{-1}h = hg^{-1}$. [6]

(c) Let G be a group, and h an element of G. Prove that

$$\{g \in G : gh = hg\}$$

is a subgroup of G. [6]

Question 8 Let the operations of addition and multiplication on the set

$$K = \{at + bu : a, b \in \mathbb{R}\},$$

where t and u are formal symbols, be defined as follows:

$$(at + bu) + (ct + du) = (a + c)t + (b + d)u,$$
$$(at + bu) \cdot (ct + du) = (ac + ad + bc - bd)t + (-ac + ad + bc + bd)u.$$

(a) Compute $(\frac{1}{2}t - \frac{1}{2}u)^2$ and express the result in the form $at + bu$. [3]

(b) Find a multiplicative identity in K, and prove that the multiplication in K satisfies the identity law. [4]

(c) Specify a bijection $f : \mathbb{C} \to K$ such that $f(\alpha + \beta) = f(\alpha) + f(\beta)$ and $f(\alpha\beta) = f(\alpha)f(\beta)$ for all complex numbers α and β. [6]

[Such a bijection is called an *isomorphism* of rings.]

End of Paper