Queen Mary
University of London

B. Sc. Examination by course unit 2014

MTH4103: Geometry I

Duration: 2 hours

Date and time: 12th May 2014, 10:00–12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Please check now to ensure you do not have any notes, mobile phones or unauthorised electronic devices on your person. If you have any, then please raise your hand and give them to an invigilator immediately. Please be aware that if you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. Disruption caused by mobile phones is also an examination offence.

Exam papers must not be removed from the examination room.

Examiner(s): J. N. Bray

Throughout, you should simplify your answers as much as possible.

Question 1 Let $A = (1,3,-1)$ and $B = (4,1,1)$, and let $\mathbf{a}$ and $\mathbf{b}$ be the position vectors of A and B respectively. Determine:

(a) the length of $\mathbf{a}$; [**2**]

(b) the vector having length 3 in the same direction as $\mathbf{a}$; [**3**]

(c) the vector represented by $\overrightarrow{AB}$; [**2**]

(d) parametric equations for the line through A and B; [**4**]

(e) the cosine of the angle between $\mathbf{a}$ and $\mathbf{b}$; [**3**]

(f) the vector product $\mathbf{a} \times \mathbf{b}$ of $\mathbf{a}$ and $\mathbf{b}$. [**4**]

Question 2 Let $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ -6 \end{pmatrix}$, and let $\mathbf{b}$ and $\mathbf{c}$ be position vectors of points in the (x,y)-plane such that the area of the parallelogram with sides $\mathbf{b}$ and $\mathbf{c}$ is 7. Let the parallelepiped with sides corresponding to $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ have volume V.

(a) Is there sufficient information to calculate V? [Answer: ‘Yes’ or ‘No’.] [**2**]

(b) (i) If your answer to Part (a) was ‘Yes’, determine V.

(ii) If your answer to Part (a) was ‘No’, specify an extra piece (or extra pieces) of information necessary and sufficient to determine V. [If you specify redundant information you will gain no marks.] [**2**]

Question 3

(a) Use Gaußian elimination (to reduce to echelon form) followed by back substitution to determine *all* solutions to the following system of linear equations in x, y, z defined over $\mathbb{R}$:

$$\left.\begin{aligned} -x - 2y + 5z &= 8 \\ 3x + 2y + z &= -4 \\ x + y - z &= -3 \end{aligned}\right\}.$$

[**8**]

(b) What does your answer to Part (a) tell you geometrically about the intersection of the three planes defined by the equations above? [**2**]

Question 4 Calculate the distance between the **parallel** lines with vector equations

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix}$$

respectively. (The parameters λ and μ both range over the whole of $\mathbb{R}$.) [**6**]

Question 5 Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & -2 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 1 & -2 & -3 \end{pmatrix}.$$

Calculate the following:

(a) $-2A + B$; [**4**]

(b) A^2; [**4**]

(c) the characteristic polynomial of B; [**5**]

(d) $\det B$ [Hint: the answer to the previous part should help]. [**3**]

Question 6 In this question, points A, B, C, D, P, Q, R and S have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$ respectively.

(a) Prove that $ABCD$ is a parallelogram if and only if $\mathbf{a} + \mathbf{c} = \mathbf{b} + \mathbf{d}$. [**3**]

(b) Now let A, B, C, D be **any** four points in 3-space, and let P, Q, R, S be the respective mid-points of the line segments AB, BC, CD, DA. Prove that $PQRS$ is a parallelogram. [**5**]

Question 7

(a) Define precisely, **without using coördinates**, the *scalar product* $\mathbf{u}\cdot\mathbf{v}$ of vectors $\mathbf{u}$ and $\mathbf{v}$. **[3]**

(b) Define precisely, **without using coördinates**, the *vector product* $\mathbf{u}\times\mathbf{v}$ of vectors $\mathbf{u}$ and $\mathbf{v}$. **[5]**

(c) Prove that for all vectors $\mathbf{a}$ and $\mathbf{b}$ we have $|\mathbf{a}\times\mathbf{b}| \leqslant |\mathbf{a}||\mathbf{b}|$. Determine precisely when equality holds. **[8]**

Question 8

(a) Define what it means for a map $t:\mathbb{R}^m \to \mathbb{R}^n$ to be a *linear transformation*. **[4]**

(b) Define what is meant by an *eigenvector* of an $n\times n$ matrix A, and the *eigenvalue* corresponding to that eigenvector. **[4]**

(c) Let S_θ denote the 2×2 matrix representing the reflexion (in the (x,y)-plane) in the line through the origin at anticlockwise angle $\theta/2$ to the x-axis. Then

$$S_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

(i) Determine the eigenvalues of S_θ. **[4]**

(ii) Now let S be the matrix of the reflexion in the line $y = x$. Write down S, and write down an eigenvector of S and its corresponding eigenvalue. **[4]**

(d) Let A be an $n\times n$ invertible matrix with eigenvalue λ. Prove that $\lambda \neq 0$, and prove that A^{-1} has λ^{-1} as an eigenvalue. **[6]**

End of Paper