Queen Mary University of London

Main Examination period 2023 – May/June – Semester B

MTH6158: Ring Theory

Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

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Question 1 [30 marks].

(a) Give an example of a non-commutative ring without an identity. **[4]**

(b) Does the equation $(1 + a)(1 - a) = 1 - a^2$ hold for any element a of a ring with identity? Explain. **[4]**

(c) Give an example of a subring of $\mathbb{Z}/14\mathbb{Z}$ having 4 elements, or explain why it does not exist. **[4]**

(d) Prove, using the axioms of a ring or the basic properties proved in the lectures, that any two elements a, b of a ring satisfy the equation $(-a)b = -(ab)$. **[6]**

(e) Give an example of a commutative ring without identity having a subring with identity, or explain why such an example cannot exist. **[6]**

(f) Explain what is wrong in the following "proof" that every finite commutative ring with identity is a field. **[6]**

"Proof": Suppose R is a finite commutative ring with identity. Let a be a non-zero element of R. We want to show that there exists an inverse of a in R, that is, an element b such that $ab = ba = 1$. Consider the set $S = \{a, a^2, a^3, \dots\}$. Since R is finite, this set S must be finite. This means that there exist positive integers $m > n$ such that $a^m = a^n$. We then have $a^{m-n} = 1$, which means that the element a^{m-n-1} is a multiplicative inverse of a. Thus every non-zero element of R has an inverse, and therefore R is a field.

Question 2 [20 marks]. Consider the ring $R = \mathbb{Z}/15\mathbb{Z}$ and its ideal $I = \{[0]_{15}, [3]_{15}, [6]_{15}, [9]_{15}, [12]_{15}\}$. [You are not required to prove that I is an ideal of R.]

(a) Is the ideal I a ring with identity? Explain. **[4]**

(b) Write down explicitly the partition of R into cosets of I. **[6]**

(c) Give an explicit isomorphism between the rings $\mathbb{Z}/3\mathbb{Z}$ and R/I. [You do not need to prove that it is an isomorphism.] **[4]**

(d) Does the equation $x^3 + x^5 + x^7 = 1$ have a solution in the ring R/I? Explain. **[6]**

Question 3 [30 marks].

(a) Give an example of a domain R and an element $a \in R$ that is neither a unit nor a zero-divisor. [**4**]

(b) For which integers $m \geq 2$ does the ring $\mathbb{Z}/m\mathbb{Z}$ satisfy the cancellative law for multiplication? Explain. [**4**]

(c) Consider the subring $S = \{a + b\sqrt{3} : a, b \in \mathbb{Z}\}$ of the ring $\mathbb{R}$ of real numbers.

(i) Explain why S is an integral domain. [**4**]

(ii) Show that the element $2 + \sqrt{3}$ is a unit of S. [**4**]

(iii) Find a factorisation of the element $6 \in S$ as a product of two elements of S that are not in $\mathbb{Z}$. [**4**]

(iv) Given that the element $6 \in S$ can also be factored as $6 = 2 \cdot 3$, can we conclude that S is not a unique factorisation domain? Explain. [**4**]

(d) Suppose R is a domain and $a \in R$ is a non-zero element satisfying $a^3 = a$. Show that a is either a unit or a zero-divisor. [**6**]

Question 4 [20 marks]. Consider the field of 2 elements $K = \mathbb{Z}/2\mathbb{Z}$ and the polynomial $f = x^3 + x + 1 \in K[x]$.

(a) Explain why f is an irreducible element of $K[x]$. [**6**]

(b) Let F be the quotient ring $F = K[x]/\langle f \rangle$, which contains the field K.

(i) Explain why F is a field. [You may use any result proved in the lectures.] [**4**]

(ii) How many elements does the field F have? [**4**]

(iii) Let α be an element of F such that $f(\alpha) = 0$. Find an expression for the inverse α^{-1} of the form $\alpha^{-1} = a \cdot \alpha^2 + b \cdot \alpha + c$ with $a, b, c \in K$. [**6**]

End of Paper.