Queen Mary
University of London

Main Examination period 2020 – May/June – Semester B
Online Alternative Assessments

MTH6112: Actuarial Financial Engineering

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment, you may use books, notes, and the internet. You may use calculators and computers, but you should show your working for any calculations you do. You must not seek or obtain help from anyone else.

At the start of your work, please **copy out and sign** the following declaration:

> I declare that my submission is entirely my own, and I have not sought or obtained help from anyone else.

All work should be **handwritten**, and should **include your student number**.

You have **24 hours** in which to complete and submit this assessment. When you have finished your work:

- scan your work, convert it to a **single PDF file** and upload this using the upload tool on the QMplus page for the module;
- e-mail a copy to maths@qmul.ac.uk with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are not expected to spend a long time working on this assessment. We expect you to spend about **2 hours** to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the assessment period, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

IFoA exemptions
This module counts towards IFoA actuarial exemptions. For your submission to be eligible for IFoA exemptions, you must submit within the first **3 hours** of the assessment period. You may then submit a second version later in the assessment period if you wish, which will count only towards your degree. There are two separate upload tools on the QMplus page to enable you to submit a second version of your work.

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Question 1 [20 marks]. Let W_t be a standard Wiener process and let $\{\mathcal{F}_t = \sigma(W_s, 0 \le s \le t)\}_{t\ge 0}$ be its natural filtration.

Q1.(a) Prove that the process $X_t = \frac{1}{2}W_{4t}$ is a Wiener process. **[4]**

Q1.(b) What does it mean for the process X_t to be a **martingale** with respect to the filtration $\{\mathcal{F}_t\}_{t\ge 0}$? **[4]**

Q1.(c) Assuming $s \le t$ derive the conditional expectations $\mathbb{E}\left[W_t^2 | \mathcal{F}_s\right]$ and $\mathbb{E}\left[W_t^4 | \mathcal{F}_s\right]$. **[8]**

Q1.(d) Using the result you obtained in **Q1.(c)** prove that $W_t^4 - 6tW_t^2 + 3t^2$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\ge 0}$. You may assume $\mathbb{E}\left[\left|W_t^4 - 6tW_t^2 + 3t^2\right|\right] < \infty$ for all $t > 0$. **[4]**

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Question 2 [15 marks]. Consider an Ito process X_t satisfying the stochastic differential equation

$$\mathrm{d}X_t = a\left(t, X_t\right)\mathrm{d}t + b\left(t, X_t\right)\mathrm{d}W_t,$$

where W_t is a Wiener process.

Q2.(a) Write down the stochastic differential equation satisfied by the process $Y_t = f\left(t, X_t\right)$. **[4]**

Q2.(b) The Ito process X_t satisfies the following stochastic equation

$$\mathrm{d}X_t = \frac{1}{2}X_t\left(1 - X_t\right)\left(1 - 2X_t\right)\mathrm{d}t - X_t\left(1 - X_t\right)\mathrm{d}W_t, \qquad \text{with } X_0 = \frac{1}{2}.$$

Using the chain rule, compute the differential $\mathrm{d}Y_t$ for the process $Y_t = \log\left(X_t^{-1} - 1\right)$. **[7]**

Q2.(c) Solve the stochastic differential equation for Y_t and thus find X_t in terms of t and W_t. **[4]**

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Question 3 [31 marks]. In this question we assume **all** assumptions underlying the Black-Scholes model are satisfied.

Q3.(a) List 3 assumptions on **the market** underlying the Black-Scholes model. Explain which one does imply existence of the risk-neutral probability distribution. **[5]**

Q3.(b) Let

$$f(t, S_t) = S_t^2 \mathrm{e}^{(r+\sigma^2)(T-t)} - \mathrm{e}^{-r(T-t)} K^2$$

be the value at time t of an option on the asset having price S_t at time t and a constant volatility σ. Let T and K be some fixed parameters of the option. Calculate corresponding option's Greeks $\Delta, \Gamma, \theta, \nu$. **[4]**

Q3.(c) What do **delta** and **gamma** hedging mean? **[4]**

Q3.(d) A put option on a stock, valued at £ 17.6, has a price £ 1.2, delta $\Delta = -0.5$ and a gamma $\Gamma = 0.1$. Another derivative on the same stock with delta $\Delta = 0.2$ and gamma $\Gamma = -0.1$ becomes available on the market at the price £ 3.52. An investor has £ 10,000 in cash which he/she is willing to invest into a delta and gamma hedged portfolio. What allocations to stock, put option and the derivative should he/she use? **[4]**

Q3.(e) For the portfolio found in **Q3.(d)** determine its θ if continuously compounded risk-free interest rate is 5%. **[2]**

Q3.(f) Give the definition of the **implied volatility** and explain how one can calculate it for a given European call option premium, strike price, maturity, stock price and interest rate. **[5]**

Q3.(g) An investor buys a European call option on a non-dividend paying stock whose current price is £ 5,000. The option premium is £ 187.06. The strike price of the call is £ 5,250 and the time to expiry is half of a year. The risk-free rate of return is 5% per annum continuously compounded. Estimate the implied volatility. You may start with two guesses $\sigma_1 = 0.1$ and $\sigma_2 = 0.2$ and then do a linear interpolation. **[7]**

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Question 4 [18 marks]. In the Vasicek model, the interest rate $r(t)$ is governed by the stochastic differential equation

$$\mathrm{d}r(t) = -a(r(t) - b)\,\mathrm{d}t + \sigma \mathrm{d}W_t, \text{ with } r(0) = r_0,$$

where W_t is the Wiener process, and $a > 0, b > 0$ are constants.

Q4.(a) Using the chain rule, derive the stochastic differential equation solved by $U(t) = \mathrm{e}^{at}(r(t) - b)$ and solve it. **[7]**

You are reminded that the solution of stochastic differential equation for $r(t)$ is

$$r(t) = b + (r(0) - b)\,\mathrm{e}^{-at} + \sigma \int_0^t \mathrm{e}^{-a(t-s)}\mathrm{d}W_s.$$

Q4.(b) State the distribution of $r(t)$ and find the probability $p_t = \mathbb{P}[r(t) < 0]$ when $t \to \infty$. **[7]**

You are reminded that a zero-coupon bond paying £1 at time T, within the context of Vasicek model, has at time $t < T$ the price equal to

$$B(t, T, r_t) = \mathrm{e}^{-A(\tau)r_t + B(\tau)},$$

where $\tau = T - t$ and

$$A(x) = \frac{1 - \mathrm{e}^{-ax}}{a}, \quad B(x) = \left(b - \frac{\sigma^2}{2a^2}\right)(A(x) - x) - \frac{\sigma^2}{4a}A^2(x),$$

and r_t is a current risk-free interest rate.

Q4.(c) Historical data of short time risk-free interest rate is given in the table for a period January-May 2020

Date	1st Jan	1st Feb	1st Mar	1st Apr	1st May
r_t	3.56%	4.02%	3.84%	4.00%	4.18%

There are three zero-coupon bonds (see the table for their parameters) available at the market paying £1 on a corresponding maturity day.

	Issue date	Maturity date	Price on issue date
Bond 1	01/01	01/03	£0.92
Bond 2	01/02	01/04	£0.86
Bond 3	01/03	01/05	?

Find the price of Bond 3. **[4]**

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Question 5 [16 marks].

Q5.(a) A company has just issued zero-coupon bonds expiring in 2 years, with a nominal value of £ 3 million. The total value of the company now stands at £ 4 million. A constant risk-free rate of return 5% per annum continuously compound is avaliable in the market. Under the Merton model and using the Black-Scholes formula find the current value of shareholder's equity, assuming that the annual volatility of the values of the company's assets is 20%. **[4]**

Q5.(b) Find the current value of the company's debt under the same Merton model and the theoretical price of £ 100 nominal of zero-coupon bonds. **[4]**

Q5.(c) Write down the distribution for the total value of the company under this Merton model. **[4]**

Q5.(d) Find the risk-neutral probability of company's bonds default. **[4]**

End of Paper – An appendix of 2 pages follows.

Properties of Gaussian random variables

A random variable ξ is said to have a Normal distribution $\mathcal{N}\left(\mu,\sigma^2\right)$ with mean μ and variance σ^2 if its probability distribution function is given by

$$f_{\xi}\left(t\right) = \frac{1}{\sqrt{2\pi\sigma^2}}\mathrm{e}^{-\frac{(t-\mu)^2}{2\sigma^2}}, \quad t \in \left(-\infty,\infty\right).$$

The cumulative distribution function of a standard Normal random variable takes a form

$$\mathbb{P}\left[\mathcal{N}\left(0,1\right) < x\right] =: \Phi\left(x\right) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{x} \mathrm{e}^{-t^2/2}\mathrm{d}t,$$

and its values are given on the next page. For a Gaussian random variable with mean μ and variance σ^2 the cumulative distribution function is given by

$$\mathbb{P}\left[\mathcal{N}\left(\mu,\sigma^2\right) < x\right] =: \Phi\left(\frac{x-\mu}{\sigma}\right).$$

Moment generating function of a Gaussian random variable $\xi = \mathcal{N}\left(\mu,\sigma^2\right)$ is equal to

$$\mathbb{E}\left[\mathrm{e}^{\xi t}\right] = \mathrm{e}^{\mu t + \frac{\sigma^2}{2}t^2}.$$

Moments of a centred Gaussian random variable $\xi = \mathcal{N}\left(0,\sigma^2\right)$ are given by

$$\mathbb{E}\left[\xi^k\right] = \begin{cases} 0, & k \text{ is odd} \\ \frac{(2p)!}{2^p p!}\sigma^{2p}, & k = 2p. \end{cases}$$

The first four moments of a non-centred Gaussian random variable $\xi = \mathcal{N}\left(\mu,\sigma^2\right)$ are equal to

$$\mathbb{E}\left[\xi\right] = \mu, \quad \mathbb{E}\left[\xi^2\right] = \mu^2+\sigma^2, \quad \mathbb{E}\left[\xi^3\right] = \mu^3+3\mu\sigma^2, \quad \mathbb{E}\left[\xi^4\right] = \mu^4+6\mu^2\sigma^2+3\sigma^4.$$

The Black-Scholes formula

Within the framework of the Black-Scholes model the price of a **European call option** with the strike price K and expiration time T at time t is given by

$$C\left(t,S_t,K,\sigma,r,T\right) = S_t\Phi\left(\omega\right) - K\mathrm{e}^{-r(T-t)}\Phi\left(\omega - \sigma\sqrt{T-t}\right),$$

where $\omega = \frac{\log\frac{S_t}{K}+r(T-t)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}$. Within the same assumptions price $f\left(t,S_t\right)$ of the derivative on the stock with price process S_t satisfies the Black-Scholes PDE

$$\frac{\partial f}{\partial t}\left(t,S_t\right) + rS_t\frac{\partial f}{\partial S_t}\left(t,S_t\right) + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial S_t^2}\left(t,S_t\right) = rf\left(t,S_t\right).$$

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Table of the cumulative standard normal distribution

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt, \quad \Phi(-x) = 1 - \Phi(x)$$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

End of Appendix.

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