Queen Mary University of London

Main Examination period 2019

MTH4115 / MTH4215: Vectors & Matrices

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: O. Jenkinson, R. Johnson

Question 1. [20 marks] Let A, B, C be points in 3-space with respective position vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix}$. Determine:

(a) The length of the vector $3\mathbf{a} - \mathbf{b}$; [3]

(b) A unit vector in the direction of $\mathbf{b}$; [3]

(c) $\mathbf{a} \cdot \mathbf{b}$; [3]

(d) $\mathbf{a} \times \mathbf{b}$; [3]

(e) A vector equation for the line through A and B; [4]

(f) The coordinates of the point D such that $ABCD$ is a parallelogram. [4]

Question 2. [20 marks] Suppose that vectors $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ are given.

(a) Write down an expression for the **scalar product** $\mathbf{u} \cdot \mathbf{v}$ (in terms of the coordinates of $\mathbf{u}$ and $\mathbf{v}$). [3]

(b) What does it mean to say that two vectors are **orthogonal**? [3]

(c) Show that if a vector is orthogonal to all vectors, then it must be the zero vector. [4]

(d) How is the **vector product** $\mathbf{u} \times \mathbf{v}$ defined (in terms of the coordinates of $\mathbf{u}$ and $\mathbf{v}$)? [3]

(e) Show that $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$. [3]

(f) Show that if $\mathbf{u}$ has the property that $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ for all vectors $\mathbf{v}$, then necessarily $\mathbf{u} = \mathbf{0}$. [4]

Question 3. [20 marks] Let Π_1 be the x-y plane (i.e. with equation $z = 0$), let Π_2 be the x-z plane (i.e. with equation $y = 0$), let Π_3 be the y-z plane (i.e. with equation $x = 0$), and let Π_4 be the plane with equation $x + y + z = 1$. Let Q be the point with position vector $\mathbf{q} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$.

(a) Determine the distance between Q and Π_1. **[2]**

(b) Determine the distance between Q and Π_4. **[3]**

(c) Determine the coordinates of the point on Π_4 that is closest to Q. **[3]**

(d) If A denotes the point in the intersection $\Pi_1 \cap \Pi_2 \cap \Pi_4$, and B denotes the point in the intersection $\Pi_1 \cap \Pi_3 \cap \Pi_4$, determine the coordinates of the mid-point C of A and B. **[3]**

(e) If l denotes the line through the points C (from part (d) above) and Q, then determine the coordinates of the point in the intersection $l \cap \Pi_3$. **[4]**

(f) Determine the coordinates of a point which is equidistant from the four planes Π_1, Π_2, Π_3, Π_4 (i.e. the point has the same distance from each of these planes). **[5]**

Question 4. [20 marks] Consider the linear system

$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 0 \\ 2x_1 & - & 3x_2 & + & 4x_3 & - & 3x_4 & = & 0 \\ -x_1 & + & x_2 & - & 3x_3 & + & 2x_4 & = & 0 \end{array} .$$

(a) Write down the augmented matrix of the system. **[3]**

(b) Bring the augmented matrix to reduced row echelon form, indicating the elementary row operations used at each step. **[4]**

(c) Identify the leading and the free variables, and write down the solution set of the system. **[4]**

(d) Let l_1, l_2 and l_3 be lines in 3-space, such that l_1 passes through $(1, 4, -3)$ in the direction $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, l_2 passes through $(1, 3, -2)$ in the direction $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$, and l_3 passes through $(2, 6, -4)$ in the direction $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.

Write down parametric equations for each of these three lines. **[3]**

(e) For the lines l_1, l_2, l_3 as in part (d) above, determine the intersection $l_1 \cap l_2$ of l_1 and l_2, the intersection $l_1 \cap l_3$ of l_1 and l_3, and the intersection $l_2 \cap l_3$ of l_2 and l_3. **[6]**

Question 5. [20 marks] Let

$$A = \begin{pmatrix} 1 & 3 \\ -2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 0 & 0 & 3 \\ 9 & 0 & 1 & 8 \\ -8 & 2 & 4 & 5 \\ 3 & 0 & 0 & 5 \end{pmatrix}.$$

(a) For each of the products A^2, AB, BA, B^2, BC, CB, state whether or not it exists; if it exists then evaluate it. **[6]**

(b) Explain what it means for a matrix M to be **invertible**, and what is meant by the **inverse** of M. **[4]**

(c) Calculate $\det(C)$ and decide whether C is invertible or not. **[4]**

(d) Using part (c) above, evaluate $\det(C^6)$ and $\det(3C)$. In each case, briefly explain which property of determinants you are using. **[4]**

(e) Find $\det(D)$, where D is the matrix obtained from C by subtracting 13 times column 1 from column 4. Briefly explain which property of determinants you are using. **[2]**

End of Paper.