Queen Mary
University of London

Main Examination period 2023 – January – Semester A

MTH4113 / MTH4213: Numbers, sets & functions

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The exam is intended to be completed within **2 hours**. However, you will have a period of **3 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

The exam is closed-book, and **no outside notes are allowed**.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any unauthorised notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: Matthew Fayers and Shahn Majid

Question 1 [20 marks].
Let

$$X = \{1, 3, 4, 6, 9\}, \qquad Y = \{2, 3, 5, 8, 9\}.$$

Write down each of the following sets. *[No justification is required.]*

(a) $X \cup Y$. **[2]**

(b) $X \triangle Y$. **[3]**

(c) $\{x \in X \ : \ x + 2 \notin X\}$. **[3]**

(d) $\{y + 2 \ : \ y \in Y$ and $y - 2 \in X\}$. **[3]**

Write down the supremum of each of the following sets. *[No justification is required.]*

(e) $\left\{x^2 \ : \ -2 \leqslant x \leqslant 1\right\}$. **[3]**

(f) $\left\{\frac{n}{n+1} \ : \ n \in \mathbb{N}\right\}$. **[3]**

(g) $\{\sin(x) \ : \ x \in \mathbb{Q}\}$. **[3]**

Question 2 [22 marks]. Suppose A and B are sets.

(a) Define precisely what it means for a function $f : A \to B$ to be **injective**. **[3]**

(b) Define precisely what it means for a function $f : A \to B$ to be **surjective**. **[3]**

Determine whether each of the following functions is injective. Justify your answers.

(c) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 20n + 22$. **[4]**

(d) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n(n + 1)$. **[4]**

(e) $f : \mathcal{P}(\mathbb{Q}) \to \mathcal{P}(\mathbb{Q})$ defined by $f(A) = A \cup \{1, 2, 3\}$. **[4]**

(f) $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z} \times \mathbb{Z}$ defined by $f(m, n) = (m^2 + n^2, m^2 - n^2)$. **[4]**

Question 3 [20 marks].

(a) Suppose P, Q and R are statements. Complete the following truth table for the statement "$(P \Rightarrow Q)$ and $(Q \Rightarrow (\text{not } R))$".

P	Q	R	$(P \Rightarrow Q)$ and $(Q \Rightarrow (\text{not } R))$
T	T	T	F
T	T	F	?
T	F	T	?
T	F	F	F
F	T	T	F
F	T	F	?
F	F	T	?
F	F	F	T

[Don't copy the whole table – just write the four missing entries in order from top to bottom in your answer booklet. You don't need to show any working.] **[4]**

(b) Suppose x, y and z are real numbers. Write down the contrapositive of the following implication.

If $x^2 > y^2$, then there is a real number w such that either $x < w$ or $w < z$. **[3]**

(c) Define a sequence $a_1, a_2, a_3, \ldots$ of integers by

$$a_1 = 0, \qquad a_n = 4a_{n-1} + 12 \text{ for } n \geqslant 2.$$

Prove by induction that $a_n = 4^n - 4$ for every $n \in \mathbb{N}$. **[8]**

(d) The following "theorem" is untrue. Explain where the mistake is in the proof. **Write no more than four sentences.**

Theorem. Suppose x is a real number satisfying

$$(x-2)^3 + 3(x-2)^2 + 2x = 4.$$

Then $x = 0$ or 1.

Proof. Let $y = x - 2$. Then the given equation becomes $y^3 + 3y^2 + 2y + 4 = 4$. Subtracting 4 from both sides and dividing through by y then gives $y^2 + 3y + 2 = 0$, which factorises to give $(y+1)(y+2) = 0$, which implies that $y = -1$ or $y = -2$. Because $x = y + 2$, this means that $x = 1$ or $x = 0$. $\square$ **[5]**

Question 4 [20 marks].

(a) Suppose d and n are natural numbers. Define what it means to say that d **divides** n. **[3]**

(b) Suppose p and q are prime numbers, and that $p \neq q$. How many divisors does $p^3 \times q^3$ have? *[No justification is required.]* **[3]**

(c) Use Euclid's algorithm to find $\gcd(198, 82)$. **[6]**

(d) Suppose $a, b \in \mathbb{N}$ and that $a \mid b$. Prove that $a^2 \mid b^2$. **[4]**

(e) Define the relation R on $\mathbb{N}$ by saying that aRb if $a \mid 2b$. Is R a transitive relation? Justify your answer. **[4]**

Question 5 [20 marks]. Suppose A and B are sets for which

$$|A| = 6, \qquad |B| = 7, \qquad |A \cap B| = 3.$$

(a) Write down the number of 2-element subsets of B. **[3]**

(b) Find $|A \cup B|$. Justify your answer briefly. **[3]**

(c) Let $D = \{C \in \mathcal{P}(A) \ : \ |C \cap B| = 1\}$. Find $|D|$. Justify your answer, using any results you need from lectures. **[4]**

(d) Let z be the complex number $1 + 3\mathrm{i}$. Find the following. *[You do not need to show your working, but doing so may help you to gain marks if you make arithmetic errors.]*

  (i) z^2. **[3]**

  (ii) $|z|^2$. **[3]**

  (iii) A complex number w such that $z - w$ and zw are both real numbers. **[4]**

End of Paper.