Main Examination period 2020 – January – Semester A

MTH4113, MTH4213: Numbers, Sets and Functions

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: Robert Johnson and Mark Walters

In this exam $\mathbb{N} = \{1, 2, 3, 4, \dots\}$.

Question 1. Let

$$A = \{1, 2, 3\}; \quad B = \{3, 4\}; \quad C = \{-1, \sqrt{2}, 4, 9\}.$$

Write down each of the following sets.

(a) $A \cup B$ **[3]**

(b) $B \triangle C$ **[3]**

(c) $C \cap \mathbb{Z}$ **[3]**

(d) $\{x \in \mathbb{R} : x + 2 \in A\}$ **[3]**

(e) The power set of B **[4]**

(f) $(A \times B) \setminus (B \times A)$ **[4]**

Question 2.

(a) State (without proof) the inclusion-exclusion formula for the cardinality of the union of two finite sets. **[3]**

(b) Suppose that X, Y and Z are subsets of $\{1, 2, 3, \dots, 10\}$ and $|X| = |Y| = |Z| = 7$.

(i) Prove that $|X \cap Y| \geq 4$. **[4]**

(ii) Deduce that $X \cap Y \cap Z$ is non-empty. **[4]**
[Hint: Consider $(X \cap Y) \cup Z$.]

(c) State (without proof) the inclusion-exclusion formula for the cardinality of the union of three finite sets. **[4]**

(d) Suppose that $|A| = |B| = |C| = 6$ and $|A \cap B| = |A \cap C| = |B \cap C| = 2$. Determine the smallest that $|A \cup B \cup C|$ can be under these conditions. Give an example of three sets which achieve this. **[5]**

Question 3.

(a) Define precisely what it means for a function $f : A \to B$ to be **injective**. **[4]**

(b) For each of the following functions, determine whether or not the function is injective justifying your answers.

(i) $t : \mathbb{Z} \to \mathbb{Z}, \quad t(n) = n - 2020$ **[4]**

(ii) $u : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N}), \quad u(S) = S \cup \{1, 2, 3\}$ **[4]**

(iii) $s : \mathbb{C} \to \mathbb{C}, \quad s(z) = z^2$ **[4]**

(iv) $p : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad p(n, m) = n + m$ **[4]**

Question 4.

(a) Suppose that $x, y \in \mathbb{R}$. Write down the negation of each of the following statements.

(i) $x + y \leq \sqrt{3}$ **[3]**

(ii) x and y are both integers. **[3]**

(iii) At least one of x and y is less than or equal to 0. **[3]**

(b) Suppose that $P(x)$ and $Q(x)$ are mathematical statements about the real number x. Write down a roadmap (as in lectures, I mean by this an outline of the structure a proof could have including the starting point and conclusion but omitting the details) for proving the following statement **using the contrapositive**. **[5]**

For all $x \in \mathbb{R}$, $P(x) \Rightarrow Q(x)$.

(c) Prove that for all $x, y \in \mathbb{R}$, if $x + y$ is irrational then at least one of x and y is irrational. Justify any properties of the rational numbers that you use. **[6]**

Question 5. Read the following passage about a number system which we did not cover in lectures.

> The idea of constructing new number systems by extending the integers is an important part of algebraic number theory. An instance of this are quadratic integer rings. We will describe an example of such a number system.
> Let $S = \{n + m\sqrt{2} : n, m \in \mathbb{Z}\}$. This set is a subset of $\mathbb{R}$ and so elements of it can be added and multiplied in the usual way. It is not hard to see that the sum of any two elements of S is an element of S. Also the product of any two elements of S is an element of S.
> Many concepts about integers have analogues in S and there are interesting similarities and differences between them.

(a) Write down two examples of elements of S and calculate their product. **[2]**

(b) Prove the assertions about the sum and product of two elements of S made in the second paragraph. **[6]**

(c) Decide whether each of the following statements is true or false giving a brief reason for each answer.

(i) $\mathbb{Z} \subseteq S$ **[3]**

(ii) $\mathbb{Q} \subseteq S$ **[3]**

(iii) $S \subseteq \mathbb{Q}$ **[3]**

(iv) If $1 = a \times b$ with $a, b \in S$ then either $a = \pm 1$ or $b = \pm 1$ **[3]**

End of Paper.