Queen Mary University of London

Main Examination period 2018

MTH4113, MTH4213: Numbers, Sets and Functions

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: R. Johnson and B. Jackson

In this exam $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$.

Question 1.

(a) Write down a description of each of the following sets in words:

(i) $\mathbb{R} \setminus \mathbb{Q}$ **[3]**

(ii) $\mathbb{Z} \setminus (\mathbb{N} \cup \{0\})$ **[3]**

(iii) $(\mathbb{Z} \setminus \mathbb{N}) \cup \{0\}$ **[2]**

(iv) $\{z \in \mathbb{C} : |z| = 1\}$ **[3]**

(b) Describe each of the following sets by listing all of their elements.

(i) The set of prime numbers which are less than 15. **[3]**

(ii) The set of complex roots of the polynomial $x^4 - 1$. **[3]**

(iii) $\{n \in \mathbb{N} : n \leq 12, \gcd(n, 6) = 1\}$. **[3]**

Question 2. Let $X = \{a, b, c\}$ and $\mathcal{P}(X)$ be the power set of X.

(a) Describe $\mathcal{P}(X)$ by listing its elements. **[4]**

(b) Decide whether each of the following statements is true or false giving a brief reason for each answer.

(i) Exactly half of the elements of $\mathcal{P}(X)$ have odd cardinality. **[4]**

(ii) X is an element of $\mathcal{P}(X)$. **[4]**

(iii) The relation R on $\mathcal{P}(X)$ defined by $A \mathrel{R} B$ if $|A \cap B| = 1$ is a transitive relation. **[4]**

(iv) The function $c : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $c(A) = X \setminus A$ is invertible. **[4]**

Question 3. Let $f : A \to B$ be a function with domain A and codomain B.

(a) Define precisely what it means for f to be **surjective**. **[4]**

(b) For each of the following functions, determine whether or not the function is surjective justifying your answers. You may assume that in each case the rule given does indeed define a function.

(i) $s : \mathbb{N} \to \mathbb{N}, \quad s(n) = n^2$. **[4]**

(ii) $c : \mathbb{C} \to \mathbb{C}, \quad c(z) = z^3 + 2z^2 + 3z + 4$. **[4]**

(iii) $t : \mathbb{Z} \to \mathbb{Z}, \quad t(n) = n + 100$. **[4]**

(iv) $p : \mathbb{N} \to \{0, 1\}, \quad p(n) = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{if } n \text{ is not prime} \end{cases}$. **[4]**

Question 4. Let $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and A and B be subsets of N.

(a) Write down the inclusion-exclusion formula for $|A \cup B|$. **[2]**

Consider the implication:

If A and B are disjoint then $|A| + |B| \leq 10$.

(b) Decide whether this implication is true or false, giving a proof or counterexample as appropriate. **[8]**

(c) Write down the converse of the implication. Is the converse true or false? Justify your answer. **[6]**

(d) Write down the contrapositive of the original implication. Is the contrapositive true or false? Justify your answer. **[4]**

Question 5. Read the following short passage which defines a property of numbers not covered in lectures and answer the questions below.

> Let n be a positive integer. The integer k is a **proper factor** of n if $1 \leq k < n$ and k divides n. The positive integer n is said to be a **perfect number** if the sum of its proper factors is equal to n. For example 6 and 28 are perfect numbers but 10 and 12 are not. Also, no prime number can be perfect.
>
> Only 50 perfect numbers are known of which the largest has more than 46 million digits. All known perfect numbers are equal to some binomial coefficient of the form $\binom{2^p}{2}$ where p is a prime. Indeed, it has been proved that every even perfect number must be of this form. It is not known whether there are infinitely many perfect numbers. It is also not known whether an odd perfect number exists.

(a) Verify that each of the numbers 6, 10, 12 and 28 has the property that the first paragraph asserts it does. **[6]**

(b) The first paragraph also makes an assertion about prime numbers. Identify and prove this assertion. **[6]**

(c) Prove that $\binom{2^p}{2}$ is even for all primes p. **[4]**

(d) Explain carefully what is wrong with the following argument: **[4]**

> The passage tells us that every even perfect number is of the form $\binom{2^p}{2}$ where p is prime. There are infinitely many prime numbers, hence there are infinitely many even perfect numbers.

End of Paper.