Queen Mary
University of London

Main Examination period 2020 – January – Semester A

MTH6154 / MTH6154P: Financial Mathematics I

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

Only non-programmable calculators that have been approved from the college list of non-programmable calculators are permitted in this examination. Please state on your answer book the name and type of machine used.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: Dr. S. Muirhead, Dr. K. Glau

Unless stated otherwise, give all numerical answers to **three significant figures**, i.e. £34.5 not £34.543 and 2.31% not 2.3092%.

Question 1 [29 marks]. Whitechapel Bank offers both deposits and loans at interest rate $r = 4\%$ that is continuously compounded.

(a) What is the effective rate offered by Whitechapel Bank? **[4]**

Whitechapel Bank also sells a 3-year bond with face value £1,000, redeemable at par, with 10% annual coupons.

(b) Write down the cash-flow stream (a_1, a_2, a_3) generated by this bond, i.e. list the payments a_i occurring at the end of years $i = 1, 2, 3$. **[3]**

(c) Using the interest rate offered by Whitechapel Bank, show that the present value of this bond is £1,164. **[4]**

(d) Explain briefly what is meant by an arbitrage opportunity. **[3]**

(e) Explain why there is an arbitrage opportunity if the issue price of the bond is less than £1,164. **[4]**

Mile End Bank offers interest that is continuously compounded at rate 8% for an introductory period of two years, after which the interest rate is continuously compounded at rate 2%.

(f) Sketch the graph of the interest rate $r(t)$ as a function of years t. **[3]**

(g) Show that the yield curve is given by the function

$$\bar{r}(t) = \begin{cases} 8\%, & \text{if } t \leq 2 \\ 2\% + \frac{12\%}{t} & \text{if } t > 2. \end{cases}$$ **[4]**

You wish to deposit £P in a bank account of one of the two banks for a duration of T years.

(h) For which values of T is it preferable to deposit the money in Mile End Bank instead of Whitechapel Bank? **[4]**

Question 2 [9 marks]. You decide to model interest rates $R(n)$ using a log-normal process with parameters $R(0) = 2\%$, $\mu = 0.2$ and $\sigma^2 = 0.2$ (here $R(n)$ is the effective interest rate for a deposit between time n and time $n+1$).

(a) Give one advantage of using a log-normal process as a model for interest rates. **[3]**

(b) Suppose that you deposit £P in a bank account at time $n = 0$. Calculate the probability that the accumulated bank balance has more than doubled by time $n = 4$. You may use the value of the standard normal c.d.f. $\Phi(1.695) = 0.955$. **[6]**

Question 3 [14 marks].
You are quoted the following market prices $V(T)$ for T-year unit zero-coupon bonds:

$$V(2) = 0.80 \quad \text{and} \quad V(4) = 0.60\,.$$

(a) Explain what is meant by a T-year unit zero-coupon bond. **[3]**

(b) Assuming no-arbitrage, calculate the spot rate s_2 and the fair forward rate $f_{2,4}$. **[4]**

You wish to sign a forward contract to buy 20 tonnes of steel in 4 years' time. The current price of steel is £220 a tonne.

(c) Explain briefly what is meant by a forward contract to buy steel. **[3]**

(d) Assuming no-arbitrage, determine the fair forward price F of this contract. **[4]**

Question 4 [15 marks]. You model a stock price $S(t)$ using a stochastic process, with t measured in years. Your model implies that the risk-neutral distribution for the stock price at $t = 2$ is

$$S(2) \overset{d}{\sim} \begin{cases} \text{£}40 & \text{with probability } 0.3, \\ \text{£}50 & \text{with probability } 0.2, \\ \text{£}75 & \text{with probability } 0.5. \end{cases}$$

Assume that interest is compounded annually at nominal rate $r = 3\%$.

(a) Explain briefly what is meant by a risk-neutral distribution. **[3]**

(b) Deduce that the current stock price is $S(0) = \text{£}56.1$. **[4]**

(c) Write down the pay-off (as a function of $S(2)$) of a European put option with strike $K = \text{£}50$ and expiry $T = 2$, and sketch the graph of this function. **[4]**

(d) Calculate the no-arbitrage price P of the put option in part (c). **[4]**

Question 5 [33 marks]. A share price is modelled via a two-period binomial model with initial stock price $S = 40$, up/down multiplication factors $u = 3/2$ and $d = 1/2$, and interest rate per time-period $r = 4\%$.

(a) Verify that the no-arbitrage assumption is valid in this model. **[4]**

(b) Show that the risk-neutral probabilities of up and down movements in the share price are
$$\tilde{p}_u = 0.54 \quad \text{and} \quad \tilde{p}_d = 0.46.$$
[4]

(c) Find the no-arbitrage price of a European call option on the share with strike $K = 60$ and expiry $T = 2$. **[9]**

(d) Use the delta-hedging formula to find the replicating portfolio $(\Psi(0), \Phi(0))$ at $t = 0$ for the option in part (c). **[6]**

Your stock broker advises you to consider buying an American call option.

(e) Explain the difference between a European and American call option. **[3]**

(f) State, without explanation, whether you would expect the price of an American call option on the share with strike $K = 60$ and expiry $T = 2$ to be higher, lower, or the same, as the corresponding European call option priced in part (c). **[3]**

The Black–Scholes formula for a European call option with strike K and expiry T written on a stock with current price S, drift μ and volatility σ is
$$C = S\Phi(\eta) - Ke^{-rT}\Phi(\eta - \sigma\sqrt{T}),$$
where
$$\eta = \frac{\ln\frac{S}{K} + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}.$$

(g) Using put-call parity, or any other method, prove that the Black-Scholes formula for a European put option with strike K and expiry T written on the same stock is
$$P = Ke^{-rT}\Phi(\sigma\sqrt{T} - \eta) - S\Phi(-\eta).$$
[4]

End of Paper.