Queen Mary
University of London

Main Examination period 2018

MTH6154: Financial Mathematics I

Duration: 2 hours

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Examiners: Dr D. S. Stark, Dr K. Glau

Question 1. [20 marks]

(a) Suppose that you put 150 pounds in a bank account which has a nominal rate of interest of 2.5% compounded semi-annually. You withdraw the money from your account five years later.

  (i) Find the effective interest rate r_{eff}. **[3]**

  (ii) How much money did you withdraw from your bank account? **[3]**

  (iii) Find the rate of return of this investment. **[4]**

(b) Suppose that you put 75 pounds in a bank account which has a nominal rate of interest of 6% compounded continuously. You never withdraw any money from your account. Find the effective interest rate r_{eff}. **[3]**

(c) Suppose that interest is compounded continuously at rate

$$r(t) = \begin{cases} at & \text{if } 0 \leq t \leq 1; \\ a & \text{if } t > 1 \end{cases}$$

where $0 < a < 1$ is a constant.

  (i) Determine the yield curve $\overline{r}(t)$ corresponding to $r(t)$. **[4]**

  (ii) What is the present value of one pound received one year from now (in other words, the discounting factor)? **[3]**

Question 2. [20 marks]

(a) Show that the present value of a perpetuity entitling its holder to be paid the constant amount c at the end of each of an infinite sequence of years is c/r if the interest rate is r compounded yearly. **[5]**

(b) Consider a four-year investment that, for an initial payment of 1000 pounds, produces the following returns: 500 pounds at the end of the second year, 750 pounds at the end of the fourth year, and nothing at the end of the first and third years. What is the internal rate of return of this investment? **[5]**

(c) Let $V_A(r)$ denote the present value of a cash-flow of assets and let $V_L(r)$ denote the present value of a cash-flow of liabilities, where r is the current continuously compounded annual interest rate. State the three conditions on $V_A(r)$ and $V_L(r)$ needed for Reddington immunisation. **[3]**

(d) Consider a bond with face value 100 pounds which pays coupons annually at coupon rate 10% and which matures in three years redeemable at par. Assume a continuously compounded annual interest rate of 8% per annum.

(i) Calculate the effective duration of this cash-flow. **[3]**

(ii) Calculate the convexity of this cash-flow. **[4]**

Question 3. [20 marks]

(a) Briefly explain what is meant by *spot rates* and *the spot rate curve*. **[3]**

(b) In the bond market of a certain country, a zero-coupon bond redeemable at par with maturity in n years is available at price P_n (as a percentage of face value). State or derive the formula for the spot rate s_n in terms of P_n. **[4]**

(c) A stochastic interest rate model assumes that for all years the annual interest rate is the random variable R which has the following probability distribution:

$$R = \begin{cases} 5.5\% \text{ with probability } 0.3 \\ 7.5\% \text{ with probability } 0.5 \\ 9.5\% \text{ with probability } 0.2 \end{cases}$$

(i) What is the expected accumulated amount by the end of the fifth year of an initial investment of 20,000 pounds? **[3]**

(ii) What is the accumulated amount by the end of the fifth year of an initial investment of 20,000 pounds at the expected rate of interest? **[4]**

(d) Find the upper quartile for the accumulated value at the end of 5 years of an initial investment of 1000 pounds, assuming that the annual interest rate in the ith year R_i is such that the random variables $1 + R_i$ are i.i.d. and log-normally distributed with parameters $\mu = 0.075$ and $\sigma^2 = 0.0025^2$. **[6]**

Question 4. [20 marks]

(a) State what is meant by a *Forward Contract* on a stock with agreed price F. Prove that if r is the nominal rate of interest compounded continuously and S is the initial value of the stock, then when $F < Se^{rt}$ there always exists an arbitrage opportunity. **[6]**

(b) Given an experiment with possible outcomes $i = 1, 2, \ldots n$ and possible wagers $j = 1, 2, \ldots, m$ and return function $r_i(j)$, state the Arbitrage Theorem. **[2]**

(c) Suppose that $n = m = 2$ and our return function is defined by $r_1(1) = 3$, $r_1(2) = -4$, $r_2(1) = 2$, $r_2(2) = -1$.

(i) Use the Arbitrage Theorem to show that there exists a portfolio $\overrightarrow{x} = (x_1, x_2)$ giving a guaranteed risk-free profit. **[4]**

(ii) Show that the portfolio $\overrightarrow{x} = (-1, 2)$ is a portfolio that actually gives a guaranteed risk-free profit. **[3]**

(d) Let $S(t)$ be the price of a share, let r be the continuous interest rate, and let $\mathbb{Q}$ be the risk-neutral distribution for $S(t)$. Prove that the expectation of $S(t)$ with respect to the distribution $\mathbb{Q}$ satisfies

$$\mathbb{E}_{\mathbb{Q}}(S(t)) = S(0)e^{rt}.$$ **[5]**

Question 5. [20 marks]

(a) Consider the one-period binomial model for which the price of a share is initially 100 and can either go up by 10% or down by 5% in the single period. The interest rate per period is 2%. Find the no-arbitrage price of a European put option with strike price 105. **[6]**

(b) Consider a call option in the multiperiod binomial model with strike price 150 pounds, initial stock price 200, and maturity time equal to 4 time periods. The interest rate is 5% per period, and the remaining model parameters are $u = 1.5$ and $d = 0.5$.

(i) Find the no-arbitrage price of the call. **[7]**

(ii) If the real-world probability of moving up is 0.6 in a time period, then what is the probability that the call option is exercised? **[4]**

(c) What are the assumptions of an i.i.d log-normal model of a financial asset? **[3]**

End of Paper – An appendix of 1 page follows.

Table of the cumulative standard normal distribution

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2}\,\mathrm{d}t.$$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

End of Appendix.