Main Examination period 2021 – January – Semester A

MTH6151: Partial Differential Equations

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: Dr. Mira Shamis, Prof. Alexander Sodin

Throughout we only consider partial differential equations in two independent variables (x, y) or (x, t).

Question 1 [18 marks].

(a) Write the most general linear first order partial differential equation in two variables (x, y). **[4]**

(b) Give the order of the following partial differential equations. Also, state whether the equations are linear or non-linear and homogeneous or inhomogeneous:

(i) $U_{xy} + U_{yyy} - \cos y \, U_y^2 = \sin x$,

(ii) $U_x + \tan^2(x^2 - y^2) \, U_{yy} + 4 = 0$,

(iii) $\sin U + U_y - U_x = 0$.

[3]

(c) Using the method of characteristics, or otherwise, find the general solution to

$$U_y + 5U_x = 0.$$

[4]

(d) Using the method of characteristics, or otherwise, find the general solution to

$$y^2 U_y - x^2 U_x = 0,$$

under the condition

$$U(x, 1) = \sin\left(1 + \frac{1}{x}\right).$$

[7]

Question 2 [14 marks].

(a) Classify, according to type (hyperbolic, elliptic, parabolic) the following equations:

(i) $3U_{xx} + 4U_{xy} - 8U_{yy} = 0.$ **[2]**

(ii) $U_{xx} - 5U_{xy} + 6U_y - 7U_x = 0.$ **[2]**

(b) Find the general solution $U(x, y)$ to the second order partial differential equation

$$5U_{xy} = 0.$$

[5]

(c) Solve the following partial differential equation

$$5U_x - U_{xy} = 0.$$

[5]

Question 3 [20 marks].

(a) D'Alembert's formula for the solution of the wave equation on $\mathbb{R}$ is given by

$$U(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

Provide a brief discussion of the meaning of the two terms in the right-hand side of the above formula. **[4]**

(b) Show by direct computation that D'Alembert's formula is a solution to the problem

$$\begin{aligned} &U_{tt} - c^2 U_{xx} = 0, \qquad x \in \mathbb{R}, \qquad t > 0, \\ &U(x,0) = f(x), \\ &U_t(x,0) = g(x). \end{aligned}$$

[6]

(c) Find the solution to the problem

$$\begin{aligned} &U_{tt} - c^2 U_{xx} = 0, \qquad x \in \mathbb{R}, \qquad t > 0, \\ &U(x,0) = 0, \\ &U_t(x,0) = \cos x. \end{aligned}$$

[6]

(d) What is the main difference between the wave equation and the heat equation in terms of the speed of propagation of information? **[4]**

Question 4 [28 marks]. Throughout this question, consider the following problem for the Laplace equation on a rectangle $\Omega \subset \mathbb{R}^2, \Omega = \{0 < x < a, 0 < y < b\}$

$$\begin{aligned} &U_{xx} + U_{yy} = 0, && (x,y) \in \Omega \\ &U(x,0) = 0, && U(x,b) = 0 \\ &U(0,y) = g(y), && U(a,y) = 0. \end{aligned}$$

(a) Following the method of separation of variables consider solutions of the form

$$U(x,y) = X(x)Y(y),$$

where X and Y are functions of a single argument. Show that X and Y satisfy the ordinary differential equations

$$\begin{aligned} X'' &= kX \\ Y'' &= -kY \end{aligned}$$

for some constant k. Moreover, show that $Y(0) = Y(b) = X(a) = 0$. **[6]**

(b) Show that the constant k obtained in (a) must be positive if $Y(y)$ is not identically 0 for $y \in [0,b]$. **[6]**

(c) Find the general solution to the ordinary differential equations in (a). **[4]**

(d) Use the conditions $Y(0) = Y(b) = 0$ to determine the value of the constant k and show that the solutions Y obtained in (c) must be of the form

$$Y(y) = \sin\left(\frac{n\pi y}{b}\right), \qquad n = 1, 2, 3, \ldots$$

Moreover, show that if $X(a) = 0$, then

$$X(x) = \sinh\left(\frac{n\pi(x-a)}{b}\right), \qquad n = 1, 2, 3, \ldots$$

[4]

(e) Use the Principle of Superposition to find the general solution to the Laplace equation on the rectangle Ω with the prescribed boundary conditions. **[4]**

(f) Briefly explain how would you solve the general problem

$$\begin{aligned} &U_{xx} + U_{yy} = 0, && (x,y) \in \Omega \\ &U(x,0) = f_1(x), && U(x,b) = f_2(x) \\ &U(0,y) = g_1(y), && U(a,y) = g_2(y). \end{aligned}$$

You may use a diagram to explain your idea. **[4]**

Question 5 [20 marks].

(a) Explain in a few words what is the heat kernel and what its relevance in the study of the heat equation. **[4]**

(b) Use the Fourier-Poisson formula

$$U(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\kappa t}} e^{-(x-y)^2/(4\kappa t)} f(y)dy$$

to compute the solution to the problem

$$U_t = \kappa U_{xx} \qquad (x,y) \in \mathbb{R}, \qquad t > 0,$$
$$U(x,0) = \begin{cases} 1 & x < 0 \\ 17 & x \geq 0 \end{cases}$$

Evaluate the limit of the solution when $t \to +\infty$. **[10]**

(c) What is the main difference one encounters when solving the heat equation on an interval by means of the method of separation of variables compared to the same procedure for the wave equation? What is the consequence of this difference in the behavior of the solutions to the two equations?

[6]

End of Paper – An appendix of 1 page follows.

The Laplacian in polar coordinates

The expression for the Laplacian for a function U on $\mathbb{R}^2$ in standard spherical coordinates (r, θ) is given by

$$\Delta U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial U^2}.$$

Orthogonality properties of the sine function

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} L/2 & \text{for} \quad n = m \\ 0 & \text{for} \quad n \neq m \end{cases}.$$

Gaussian integral

$$\int_0^\infty e^{-s^2} ds = \frac{\sqrt{\pi}}{2}.$$

D'Alembert's formula

$$U(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds,$$

where

$$U(x,0) = f(x), \qquad U_t(x,0) = g(x).$$

The Fourier-Poisson formula

$$U(x,t) = \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4\varkappa t}}{\sqrt{4\varkappa\pi t}} f(y)dy.$$

End of Appendix.