Queen Mary
University of London

Main Examination period 2021 – January – Semester A

MTH6151: Partial Differential Equations

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are expected to spend about **2 hours** to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: Dr. Juan A. Valiente Kroon, Dr. Arick Shao

Throughout we only consider partial differential equations in two independent variables (x, y) or (x, t).

Question 1 [25 marks].

(a) Consider a first order linear partial differential equation. Briefly explain:

(i) What happens when the characteristic curves of the equation do not cover the whole plane? **[4]**

(ii) What happens when two (or more) characteristics intersect? **[4]**

(b) State whether the following equations are linear or non-linear:

(i)

$$U_x + e^y U_y = 0,$$

[2]

(ii)

$$\frac{U_x}{1+U_x^2} + \frac{U_y}{1+U_y^2} = 0.$$

[2]

(c) Find the solution to

$$U_x - 2U_t = 0, \qquad U(0,t) = \tanh t.$$

[5]

(d) (i) Using the method of characteristics, or otherwise, find the general solution to

$$xU_x + yU_y = kU + 1, \qquad k \neq 0 \quad \text{a constant}$$

for $x > 0$. **[6]**

(ii) Make a sketch of the characteristic curves. **[2]**

Question 2 [14 marks].

(a) Given the second order partial differential equation with constant coefficients

$$aU_{xx} - aU_{xy} - cU_x + dU_y = f,$$

when is it elliptic? **[2]**

(b) Can the equation

$$aU_{xy} + bU_x + cU_y + U = 0, \qquad a \neq 0,$$

with $a,\ b,\ c \in \mathbb{R}$ constants be parabolic? **[2]**

(c) Given the problem

$$\begin{aligned} &U_{tt} - c^2 U_{xx} = 0, \qquad x \in [0, L], \qquad t \geq 0, \\ &U(x, 0) = f(x), \\ &U_t(x, 0) = g(x), \\ &U_x(0, t) = U_x(L, t) = 0, \end{aligned}$$

for the wave equation on an interval show that

$$\int_0^L (U_t^2 + c^2 U_x^2) dx$$

is a conserved quantity. **[6]**

(d) Is the quantity in (c) still conserved if one replaces the boundary conditions by

$$U(0, t) = a, \qquad U(L, t) = b,$$

where a and b are two constants? **[4]**

Question 3 [18 marks].

(a) Briefly explain what is the difference between the following two solutions to the 1+1 wave equations:

$$\begin{aligned} U(x,t) &= \frac{1}{2}\big(f(x+ct)+f(x-ct)\big)+\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds, \\ U(x,t) &= F(x-ct)+G(x+ct), \end{aligned}$$

where $f,\ g,\ F$ and G are arbitrary functions of one variable. **[4]**

(b) Show by a direct computation that

$$U(x,t)=\int_{x-ct}^{x+ct} g(s)ds$$

is a solution to the wave equation $U_{tt}-c^2U_{xx}=0$. **[4]**

(c) Let $U(x,t)$ denote a solution to the wave equation

$$U_{tt}-c^2U_{xx}=0.$$

Show that

$$V(x,t)\equiv U(x+\alpha,t+\beta)$$

is also a solution to the wave equation for any constants α and β. **[4]**

(d) Find the solution to the problem

$$\begin{aligned} &U_{tt}-c^2U_{xx}=0, \qquad x\in\mathbb{R}, \\ &U(x,0)=0, \\ &U_t(x,0)=\frac{1}{1+x^2}. \end{aligned}$$

Provide a sketch of the solution for different times. **[6]**

Question 4 [26 marks].
Throughout this question, consider the following problem for the Laplace equation on an annular region:

$$\begin{aligned}
&\Delta U = 0, \qquad (r,\theta) \in \Omega = \{a \le r \le b,\ \theta \in [0,2\pi)\},\\
&U(a,\theta) = f(\theta), \qquad U(b,\theta) = g(\theta),
\end{aligned}$$

where f and g are functions of a single argument.

(a) What does the **principle of the maximum** says about the solution to the above problem if

$$f(\theta) = 1, \qquad g(\theta) = 2?$$

Provide a brief explanation. **[4]**

(b) What happens in the above problem if

$$f(\theta) = g(\theta) = 1?$$

Provide a brief explanation. **[4]**

(c) Following the method of separation of variables consider solutions of the form

$$U(r,\theta) = R(r)\Theta(\theta)$$

where R and Θ are functions of a single argument. Obtain the ordinary differential equations satisfied by R and Θ. **[4]**

(d) Which sign should the separation constant have in order to obtain solutions which are periodic on θ? **[2]**

(e) Which value of the separation constant is consistent with the boundary conditions

$$f(\theta) = 1, \qquad g(\theta) = 2?$$

[2]

(f) Obtain the values of $R(r)$ and $\Theta(\theta)$ in the case that the separation constant is equal to zero and $\Theta(\theta)$ is periodic. **[5]**

(g) Use the results in (c), (d), (e) and (f) to obtain the solution to the problem

$$\begin{aligned}
&\Delta U = 0, \qquad (r,\theta) \in \Omega = \{a \le r \le b,\ \theta \in [0,2\pi)\},\\
&U(a,\theta) = 1, \qquad U(b,\theta) = 2.
\end{aligned}$$

[5]

Question 5 [17 marks].

(a) Assuming that $X(x) \neq 0$, prove that for the eigenvalue problem

$$\begin{aligned} &X''(x) = kX(x), \\ &X(-a) = X(a), \qquad X'(-a) = X'(a) \end{aligned}$$

one necessarily has that $k \leq 0$. **[6]**

(b) Use the Fourier-Poisson formula to compute the solution to the problem

$$\begin{aligned} &U_t = \varkappa U_{xx}, \qquad x \in \mathbb{R}, \quad t > 0, \\ &U(x,0) = 1. \end{aligned}$$

Provide an interpretation of the result you obtain. **[6]**

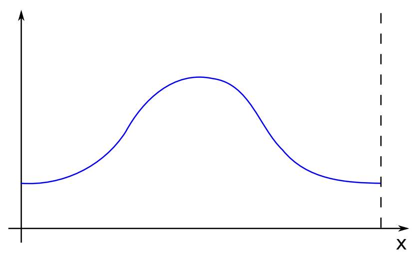
(c) Describe in qualitative terms the behaviour of the solution to the heat equation on an interval

$$U_t = \varkappa U_{xx}, \qquad x \in [0, 2\pi],$$

with initial data

$$U(x,0) = f(x)$$

where $f(x)$ has the form



and

$$U(0,t) = U(2\pi,t) = 1.$$

What do you expect to be the limit of $U(x,t)$ as $t \to \infty$? **[5]**

End of Paper – An appendix of 1 page follows.

The Laplacian in polar coordinates

The expression for the Laplacian for a function U on $\mathbb{R}^2$ in standard spherical coordinates (r, θ) is given by

$$\Delta U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}.$$

Orthogonality properties of the sine function

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} L/2 & \text{for} \quad n = m \\ 0 & \text{for} \quad n \neq m \end{cases}.$$

Gaussian integral

$$\int_0^\infty e^{-s^2} ds = \frac{\sqrt{\pi}}{2}.$$

D'Alembert's formula

$$U(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds,$$

where

$$U(x,0) = f(x), \qquad U_t(x,0) = g(x).$$

The Fourier-Poisson formula

$$U(x,t) = \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4\varkappa t}}{\sqrt{4\varkappa\pi t}} f(y)dy.$$

Other

$$\int \frac{dx}{1+x^2} = \arctan x.$$

End of Appendix.