Queen Mary University of London

Main Examination period 2020 – January – Semester A

MTH6151: Partial Differential Equations

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any unauthorised notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: Dr. Juan A. Valiente Kroon

Throughout we only consider partial differential equations in two independent variables (x, y) or (x, t).

Question 1 [18 marks].

(a) Write the most general linear second order partial differential equation in two variables (x, y). **[3]**

(b) Give the order of the following partial differential equations. Also, state whether the equations are linear or non-linear and homogeneous or inhomogeneous:

(i) $U_{tt} - U_{xxx} + U^2 = \tan x$, **[2]**

(ii) $U_t + \cos x\, U_{xtt} + \tan x = 0$. **[2]**

(c) Using the method of characteristics, or otherwise, to find the general solution to

$$\pi U_x + U_y = 0, \qquad U(x, 0) = x^2.$$

[4]

(d) Using the method of characteristics, or otherwise, to find the general solution to

$$x^2 U_x + y^2 U_y = (x + y)U.$$

[7]

Question 2 [16 marks].

(a) Classify, according to type (hyperbolic, elliptic, parabolic) the equations:

(i) $U_{xy} = 0.$ **[2]**

(ii) $U_t - 3U_{xx} + 5U = x^2.$ **[2]**

(b) Briefly explain what is understood by a **conserved quantity** of a solution to the heat equation. **[3]**

(c) Given the problem

$$\begin{aligned} &U_t = \varkappa U_{xx}, \qquad x \in [0, L], \qquad t \geq 0, \\ &U(x, 0) = f(x), \\ &U_x(0, t) = U_x(L, t) = 0, \end{aligned}$$

for the heat equation on an interval show that

$$\int_0^L U(x, t) dx$$

is a conserved quantity. Provide an interpretation. **[6]**

(d) What happens in the previous problem if one replaces the boundary conditions by

$$U_x(0, t) = a, \qquad U_x(L, t) = b,$$

where a and b are two constants? Provide an interpretation. **[3]**

Question 3 [24 marks].

(a) Explain the relevance of D'Alembert's formula

$$U(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

in the study of solutions to the wave equation. **[3]**

(b) In the problem

$$\begin{aligned} &U_{tt} - c^2 U_{xx} = 0, \qquad x \geq 0, \qquad t > 0, \\ &U(0,t) = 0, \\ &U(x,0) = f(x), \qquad U_t(x,0) = g(x), \end{aligned}$$

state the initial conditions and the boundary conditions. What sort of situation is described by the above problem? **[4]**

(c) Given a function $f(x)$ defined only for $x \geq 0$, explain what is understood by its **odd extension**. **[3]**

(d) Use D'Alembert's formula and odd extensions of functions to obtain the solution to the problem in part (b). **[5]**

(e) Verify explicitly that the solution you have obtained satisfies the right boundary conditions. **[5]**

(f) What is the interpretation of the solution you have obtained? You may use a drawing to help with your explanation. **[4]**

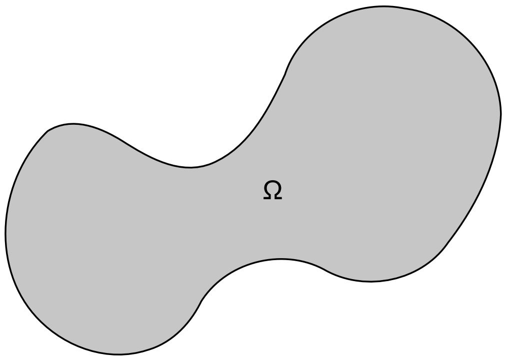
Question 4 [16 marks]. In this question consider the Laplace equation on a domain $\Omega \subset \mathbb{R}^2$.

(a) State the principle of the maximum/minimum for the Laplace equation. **[3]**

(b) Give the solution to the problem

$$\Delta U = 0, \quad \text{on} \quad \Omega,$$
$$U|_{\partial\Omega} = 1,$$

where $\Omega \subset \mathbb{R}^2$ is a domain of the form



[4]

(c) Show that the solution to the problem

$$\Delta U = f(x,y), \quad \text{on} \quad \Omega,$$
$$U|_{\partial\Omega} = g(x,y),$$

with Ω as in part (b) is unique. **[5]**

(d) Show that if $U(x,y)$ is a solution to the Laplace equation then $U(\alpha x, \alpha y)$ with $\alpha \in \mathbb{R}$ a constant is also a solution to the Laplace equation. **[4]**

Question 5 [26 marks].
Throughout this question, consider the problem

$$\begin{aligned} &U_t - \varkappa U_{xx} = 0, \qquad x \in [0, L], \quad t \geq 0, \\ &U(x, 0) = f(x), \\ &U(0, t) = 0, \qquad U(L, t) = 0. \end{aligned} \tag{1}$$

(a) Following the method of separation of variables consider solutions of the form

$$U(x, t) = X(x)T(t)$$

where X and T are functions of a single variable. Show that X and T satisfy the ordinary differential equations

$$\begin{aligned} X'' &= kX, \\ T' &= \varkappa kT, \end{aligned}$$

for some constant k. Moreover, show that

$$X(0) = X(L) = 0.$$

[6]

(b) Show that the constant k obtained in (a) must be negative. **[5]**

(c) Find the general solution to the ordinary differential equations in (a). **[4]**

(d) Use the conditions $X(0) = X(L) = 0$ to determine the value of k and show that the solutions X obtained in (c) must be of the form

$$X(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, 3, \ldots$$

[4]

(e) Use the **Principle of Superposition** to find the general solution to the heat equation (1) on the interval $[0, L]$ with the prescribed boundary conditions. **[4]**

(f) Find the particular solution corresponding to the intial data

$$U(x, 0) = \sin\left(\frac{3\pi x}{L}\right) + 7\sin\left(\frac{6\pi x}{L}\right).$$

[3]

End of Paper – An appendix of 1 page follows.

The Laplacian in polar coordinates

The expression for the Laplacian for a function U on $\mathbb{R}^2$ in standard spherical coordinates (r, θ) is given by

$$\Delta U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial U^2}.$$

Orthogonality properties of the sine function

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} L/2 & \text{for} \quad n = m \\ 0 & \text{for} \quad n \neq m \end{cases} .$$

Gaussian integral

$$\int_0^\infty e^{-s^2} ds = \frac{\sqrt{\pi}}{2}.$$

D'Alembert's formula

$$U(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s) ds,$$

where

$$U(x,0) = f(x), \qquad U_t(x,0) = g(x).$$

The Fourier-Poisson formula

$$U(x,t) = \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4\varkappa t}}{\sqrt{4\varkappa\pi t}} f(y) dy.$$

End of Appendix.