Queen Mary University of London

Main Examination period 2019

MTH6909: Bayesian Statistics

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Only non-programmable calculators that have been approved from the college list of non-programmable calculators are permitted in this examination. Please state on your answer book the name and type of machine used.

Statistical functions provided by the calculator may be used provided that you state clearly where you have used them.
The New Cambridge Statistical Tables 2nd Eddition are provided.
A table of common distributions is provided as an appendix.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiners: L I Pettit, J Griffin

Question 1. [20 marks]

(a) Show that a Poisson distribution with mean θ is an exponential family. [**6**]

(b) If a random sample of n Poisson distributed observations is collected write down a sufficient statistic for θ. [**2**]

(c) It is believed that the number of accidents in a new factory will follow a Poisson distribution with mean θ per month. The prior distribution of θ is given by a gamma distribution $Ga(\alpha, \beta)$.

(i) A safety inspector assesses that based on his experience at a similar factory $\alpha = 12, \beta = 4$. If there are 18 accidents in the first eight months, derive the posterior distribution of θ and find its mean and variance. [**7**]

(ii) Show the posterior mean can be written as a weighted average of the prior mean and the sample mean. [**5**]

Question 2. [15 marks] A horticulturalist is interested in the probability θ that a seed of a particular variety germinates successfully. Her prior distibution for the germination probability can be represented by the $Be(a, b)$ distribution. In an experiment she sows n seeds of which x germinate successfully.

(a) Find her posterior distribution for θ. [**4**]

(b) If she decides that $a = 3, b = 1$ and she observes $x = 7$ successes with $n = 10$ seeds, find her posterior distribution for θ. [**2**]

(c) If she wishes to estimate θ using a quadratic loss function

$$l(t, \theta) = (t - \theta)^2$$

derive the Bayes estimate of θ and the expected loss and calculate their values. [**9**]

Question 3. [25 marks]

(a) A random sample of failure times $t_1, \ldots, t_n$ is observed for n machine components. Each t_i is assumed to have an exponential distribution with mean λ^{-1} and the prior distribution for λ is taken as $Ga(\alpha, \beta)$ with parameters α and β. Find the posterior distribution of λ. [**5**]

(b) Show that for the situation described in (a), the marginal likelihood (or prior predictive density) is given by

$$p(t_1, \ldots, t_n) = \frac{\beta^\alpha \Gamma(\alpha + n)}{\Gamma(\alpha)(\beta + S_n)^{\alpha+n}}$$

where $S_n = \sum_{i=1}^n t_i$. [**8**]

(c) Suppose $n = 10$, $\sum t_i = 50$, $\alpha = 5$ and $\beta = 20$. Find the Bayes factor to test the null hypothesis that $\lambda = \alpha/\beta$ against an alternative that λ has a $Ga(\alpha, \beta)$ prior. What is your conclusion? [**12**]

Question 4. [15 marks]

(a) Define a $(1-\alpha)\times 100\%$ **credible interval**. [**3**]

(b) The plot below shows the density of the posterior distribution of parameter θ. Comment on what this shows. [**5**]

(c) Make a sketch of this posterior and include an example of a $100(1-\alpha)\%$ highest posterior density interval, this need not be to scale but should show the necessary properties such an interval has. [**4**]

(d) What is the advantage of a credible interval over a classical confidence interval? [**3**]

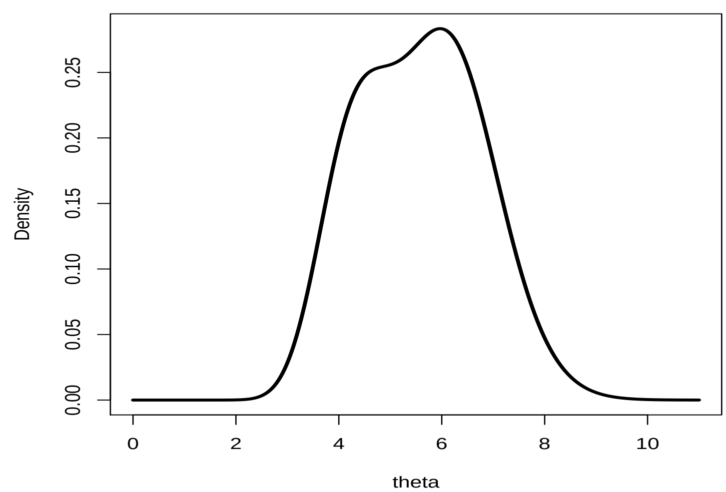

Figure 1: Plot of posterior density.

Question 5. [25 marks]
Observations are taken from a model

$$y_{ij} \sim No(\alpha_i, \xi) \qquad \text{for } i = 1, 2 \text{ and } j = 1, \ldots, n_i.$$

Note that ξ is the precision. The priors for α_1 and α_2 are respectively $No(2, 2\xi)$ and $No(6, 4\xi)$ and they are assumed independent. The prior for ξ is $Ga(a/2, b/2)$.

(a) Find the posterior distributions of α_1 and α_2 given ξ, $p(\alpha_1|\underline{y}, \xi)$ and $p(\alpha_2|\underline{y}, \xi)$. **[11]**

(b) Find the posterior distribution of ξ given α_1 and α_2, $p(\xi|\underline{y}, \alpha_1, \alpha_2)$. **[7]**

(c) Explain how samples from the unconditional posterior distributions of α_1, α_2 and ξ can be simulated using Gibbs sampling. **[7]**

Hint: For a two-stage linear model

$$\begin{aligned} \underline{y}|\underline{\theta}_1 &\sim N(A_1\underline{\theta}_1, C_1) \\ \underline{\theta}_1 &\sim N(\underline{\mu}, C_2) \end{aligned}$$

where A_1, C_1, C_2 and $\underline{\mu}$ are known, the posterior distribution of $\underline{\theta}_1$ is $N(B\underline{b}, B)$ where

$$\begin{aligned} B^{-1} &= A_1^T C_1^{-1} A_1 + C_2^{-1}, \\ \underline{b} &= A_1^T C_1^{-1} \underline{y} + C_2^{-1} \underline{\mu}. \end{aligned}$$

End of Paper – An appendix of 2 pages follows.

Bayesian Statistics – Common Distributions

Discrete Distributions

Distribution	Density	Range of Variates	Mean	Variance
Uniform	$\frac{1}{N}$	$N = 1, 2, \ldots$ $x = 1, 2, \ldots, N$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$
Bernoulli	$p^x(1-p)^{1-x}$	$0 \leq p \leq 1,\ x = 0, 1$	p	$p(1-p)$
Binomial	$\binom{n}{x}p^x(1-p)^{n-x}$	$0 \leq p \leq 1,\ n = 1, 2, \ldots$ $x = 0, 1, \ldots n$	np	$np(1-p)$
Poisson	$\frac{\exp(-\lambda)\lambda^x}{x!}$	$\lambda > 0,\ x = 0, 1, 2, \ldots$	λ	λ
Geometric	$p(1-p)^x$	$0 < p \leq 1,\ x = 0, 1, 2, \ldots$	$\frac{(1-p)}{p}$	$\frac{(1-p)}{p^2}$
Negative Binomial	$\binom{r+x-1}{x}p^r(1-p)^x$	$0 < p \leq 1,\ r > 0$ $x = 0, 1, 2, \ldots$	$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$

Continuous Distributions

Distribution	Density	Range of Variates	Mean	Variance
Uniform	$\frac{1}{b-a}$	$-\infty < a < b < \infty$ $a < x < b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal $N(\mu, \sigma^2)$	$\frac{1}{\sqrt{2\pi\sigma^2}}\exp[\frac{-(x-\mu)^2}{2\sigma^2}]$	$-\infty < \mu < \infty$ $\sigma > 0,\ -\infty < x < \infty$	μ	σ^2
Normal $No(\mu, h)$	$\frac{\sqrt{h}}{\sqrt{2\pi}}\exp[\frac{-h(x-\mu)^2}{2}]$	$-\infty < \mu < \infty$ $h > 0,\ -\infty < x < \infty$	μ	h^{-1}
Exponential	$\lambda\exp(-\lambda x)$	$\lambda > 0,\ x \geq 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma $Ga(\alpha, \beta)$	$\frac{\beta^\alpha x^{\alpha-1}\exp(-\beta x)}{\Gamma(\alpha)}$	$\alpha > 0,\ \beta > 0,\ x > 0$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$

Distribution	Density	Range of Variates	Mean	Variance
Beta $Be(a,b)$	$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}$	$a>0,\ b>0,\ 0<x<1$	$\frac{a}{a+b}$	$\frac{ab}{(a+b+1)(a+b)^2}$
$t_\nu(m,g)$	$\frac{g^{1/2}\Gamma((\nu+1)/2)}{\sqrt{(\nu\pi)}\Gamma(\nu/2)}$	$-\infty<x<\infty$	location m	precision g,
	$\times\left[1+\frac{g}{\nu}(x-m)^2\right]^{-(\nu+1)/2}$	dof ν		
F_n^m	$\frac{\Gamma[(m+n)/2]}{\Gamma(m/2)\Gamma(n/2)}\left(\frac{m}{n}\right)^{\frac{m}{2}}$	$m,n=1,2,\ldots$	$\frac{n}{n-2}$	$\frac{2n^2(m+n-2)}{m(n-2)^2(n-4)}$
	$\times\frac{x^{(m-2)/2}}{[1+(m/n)x]^{(m+n)/2}}$	$x\geq 0$	for $n>2$	for $n>4$
χ_k^2	$\frac{1}{\Gamma(k/2)2^{k/2}}x^{k/2-1}\exp(-\frac{x}{2})$	$k=1,2,\ldots,\ x>0$	k	$2k$
Pareto	$\frac{\alpha\beta^\alpha}{x^{\alpha+1}}$	$\alpha>0,\beta>0,x>\beta$	$\frac{\beta\alpha}{(\alpha-1)}$	$\frac{\beta^2\alpha}{(\alpha-1)^2(\alpha-2)}$

End of Appendix.