Main Examination period 2018

MTH751U / MTH751P / MTHM751: Processes on Networks

Duration: 3 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Question 1. [35 marks]

Avalanches on a Cayley tree.
Consider an infinite Cayley tree with branching ratio z.
Consider the following branching process:

- at time $t = 1$ the root of the Cayley tree topples with probability p;
- at every time $t > 1$, each node connected to a node that has toppled at time $t - 1$ topples with probability p.

We indicate with $\mathcal{P}(s)$ the probability that an avalanche has finite size s.
We indicate with $\pi(s)$ the probability that if we follow a link connecting a node that has toppled at time t to a node that can topple at time $t + 1$, we reach a finite subavalanche of finite size s.

a) Find a recursive equation for $\pi(s)$. Use the fact that the size of an avalanche s started from a toppling node is given by $s = 1 + \sum_{n=1}^{z} s_n$. Here s_n are the sizes of the causally connected subavalanches reached by following each of the z possible links $n = 1, 2, \ldots, z$ of the toppling node in the direction of the propagation of the avalanche. **[9]**

b) By using the properties of the generating functions show that $H_1(x)$, the generating function of $\pi(s)$, satisfies the following equation,

$$H_1(x) = 1 - p + px \left[H_1(x)\right]^z .$$

[10]

c) Show that

$$\pi(s) = P(s).$$

[5]

d) Derive the equation for the probability S that a toppling event gives rise to an infinite avalanche. **[5]**

e) Assume $z = 2$. Consider the following three possible values of the toppling probability p: $p = 0.4$ (case A), $p = 0.2$ (case B) and $p = 0.8$ (case C). In which of the cases above we do have $S > 0$? **[6]**

Question 2. [45 marks]

The SIR model on complex networks.
Consider the SIR model on a complex network, where β is the rate at which a susceptible individual in contact with an infected individual becomes infected, and μ is the rate at which an infected individual becomes removed.

a) Show that the probability density function $P(\tau)$ of the time τ required for an infected individual to become removed is given by

$$P(\tau) = \mu e^{-\mu\tau}.$$

[8]

b) The transmissibility T is given by the probability that an infected node transmits the infection to a nearest neighbour in the susceptible state. Show that the transmissibility T can be written as:

$$T = 1 - \int d\tau P(\tau) e^{-\beta\tau} = \frac{\lambda}{1+\lambda}$$

where $\lambda = \beta/\mu$. **[8]**

c) Map the SIR model on a network to the percolation process on the same network, by identifying the transmissibility T of the SIR model with the probability p that a random node is not damaged in the percolation transition. Show that the value λ_c is given by

$$\lambda_c = \frac{\langle k \rangle}{\langle k^2 \rangle - 2\langle k \rangle}.$$

[6]

d) Evaluate the epidemic threshold for: a regular network of degree distribution $P(k) = \delta_{k,6}$ (case A), a regular network of degree distribution $P(k) = \delta_{k,5}$ (case B). **[4]**

e) Evaluate the epidemic threshold for: a Poisson network of average degree $\langle k \rangle = 6$ (case C) and a Poisson network of average degree $\langle k \rangle = 5$ (case D). **[4]**

f) Consider an uncorrelated scale-free network with power-law degree distribution $P(k) = Ck^{-\gamma}$, $\gamma = 2.2$ and $k \in [1, \sqrt{N}]$. Evaluate $\langle k \rangle$ and $\langle k^2 \rangle$ in the continuous approximation. **[10]**

g) Consider a SIR epidemic spreading process on the scale-free network of point f). What is the value of the epidemic threshold λ_c in the limit $N \to \infty$? **[5]**

Question 3. [20 marks]

Random walk

Consider an undirected network of N nodes formed by a single connected component.
Indicate with i (or j or r) the generic node of the network with $i = 1, 2, \ldots, N$.
Indicate with $\mathbf{a}$ the $N \times N$ adjacency matrix of the network.
Indicate with k_i the degree of node i.
Assume that the random walks taking place on the network have a steady state.

a) Determine the steady state probability μ_i that asymptotically in time a random walker is found on node i when:

   i) the probability P_{ji} that the random walker hops from node j to node i is given by
   $$P_{ji} = \frac{a_{ji}}{k_j};$$
   [5]

   ii) the probability P_{ji} that the random walker hops from node j to node i is given by
   $$P_{ji} = \frac{a_{ji} k_i}{\sum_{r=1}^{N} k_r a_{jr}};$$
   [5]

   iii) the probability P_{ji} that the random walker hops from node j to node i is given by
   $$P_{ji} = \frac{a_{ji} (k_i)^{-\beta}}{\sum_{r=1}^{N} a_{jr} (k_r)^{-\beta}},$$
   where $\beta = 5/2$. **[5]**

b) Assume that the random walks defined in point a) take place on a random uncorrelated network in which there are two nodes (node A and node B) of degree respectivelly $k_A = 5$ and $k_B = 2$. Determine in which of the random walks studied in point a) the probability that the random walker is on node A is higher than the probability that it is on node B.
Provide your answer in the framework of the annealed approximation in which it is allowed to make the substitution
$$a_{ij} \to \frac{k_i k_j}{\sum_{r=1}^{N} k_r}.$$
[5]

End of Paper.