Queen Mary
University of London

M. Sci. Examination by course unit 2015

MTH750U: Graphs and Networks

Duration: 3 hours

Date and time: 8th May 2015, 10:00–13:00

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You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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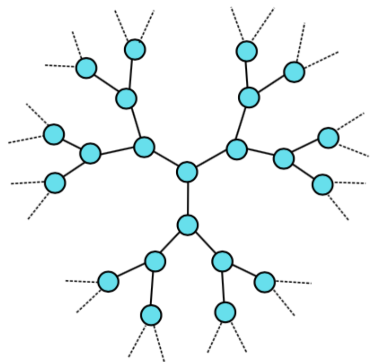
Examiner(s): Prof. Vito Latora

Question 1 (33 marks).
A Cayley tree is an infinite tree in which each node is connected to $z > 1$ neighbours, where z is called the coordination number. To construct a Cayley tree we can start with an origin node and connect this to z nodes. Each of these nodes is then connected to $z - 1$ new nodes, and this procedure is repeated infinitely often. An example of a Cayley tree with coordination number $z = 3$ is shown in the figure below.



(a) State the definitions of the clustering coefficient C and of the transitivity T of a graph. What are the values of C and T in a Cayley tree with coordination number $z = 3$? **[9]**

(b) Consider a Cayley tree with $z = 4$. How many nodes have respectively distance $d = 1, 2, 3$ from the origin? **[3]**

(c) For the general case of a Cayley tree with coordination number $z > 1$, find an expression for N_d, the number of nodes at distance d from any given node, as a function of d and z. **[4]**

(d) Consider the finite graph induced by a given node of a Cayley tree with coordination number $z > 1$, and by all the nodes of distance less than or equal to S from the first node. Find an expression for the number of nodes N in such a graph as a function of S and z. **[7]**

(e) Consider the same graph as in part (d). In the case $z = 3$, find an expression for the diameter D of such a graph as a function of the number of nodes N in the graph. Does this graph exhibit small-world behaviour? **[10]**

Question 2 (33 marks).

Consider a scale-free network with N nodes. Suppose the degree distribution is $p(k) = ck^{-\gamma}$ with exponent $\gamma > 1$, and the smallest and largest degree are respectively equal to $k_{\min}$ and $k_{\max}$. In the following, work in the so-called continuous-k approximation, i.e. treat the degree k as a real positive number.

(a) Determine the value of the normalisation constant c. [**4**]

(b) Find an expression for the average node degree, $\langle k \rangle$, and an expression for the second order moment of the degree distribution, $\langle k^2 \rangle$. [**8**]

(c) What are the values of the average degree of a node, and of the average degree of its neighbours in the case in which $k_{\min} = 1$, $k_{\max} = 1000$, and $\gamma = 2.5$? [**6**]

(d) Assume now that $k_{\min} = 1$ and $k_{\max} = \min(N^{1/(\gamma-1)}, \sqrt{N})$. Find an expression for $\langle k \rangle$ and $\langle k^2 \rangle$ when $N \to \infty$. Consider separately the three following cases: $\gamma \in (1, 2]$, $\gamma \in (2, 3]$, and $\gamma > 3$. [**9**]

(e) State the Molloy-Reed criterion. When $\gamma \in (2, 3]$, does the network considered in part (d) have a giant component in the limit $N \to \infty$? [**6**]

Question 3 (34 marks).
Consider the following model to grow graphs.
Given three positive integers $N \gg 1$, $n_0 = 5$ and $m = 3$, and a positive real number α, the graph grows, starting at time $t = 0$ with a complete graph with n_0 nodes, and by iteratively repeating at time $t = 1, 2, 3, \ldots, N - n_0$, the two steps:

(1) A new node, labeled by the index n, being $n = n_0 + t$, is added to the graph. The node arrives together with m edges.

(2) The m edges link the new node to m different nodes already present in the system. The probability $\Pi_{n\to i}$ that a new edge links the new node n to node i (with $i = 1, 2, \cdots, n-1$) is:

$$\Pi_{n\to i} = \frac{k_{i,t-1}^{\alpha}}{\sum_{l=1}^{n-1} k_{l,t-1}^{\alpha}}$$

where $k_{i,t}$ is the degree of node i at time t.

(a) Find an expression for the number of nodes, n_t, and the number of links, l_t, as a function of time t. **[5]**

(b) What is the final number of nodes and links in the graph, and what is the average node degree $\langle k \rangle$ when $N \to \infty$? **[7]**

(c) Write down the rate equations of the model, i.e. the equations for $\overline{n}_{k,t}$, where $\overline{n}_{k,t}$ denotes the average number of nodes with degree k ($k \geqslant m$) present in the graph at time t. The average, as usual, is performed over infinite realizations of the growth process with the same parameters N, n_0, m and α. **[6]**

(d) Solve the rate equations, in the case $\alpha = 1$, to find the stationary degree distribution p_k. Notice that p_k is the limit of $p_{k,t} = \overline{n}_{k,t}/n_t$ when $t \to \infty$. **[11]**

(e) For $\alpha = 1$, does the model produce scale-free networks? If so, what is the value of the degree exponent γ? **[5]**

End of Paper.