Main Examination period 2018

MTH744U/MTH744P: Dynamical Systems

Duration: 3 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: C. Joyner, F. Vivaldi

Question 1. [28 marks] (One-dimensional systems)

(a) Consider the following one-dimensional system

$$\dot{x} = x^2 - 1.$$

Find the fixed points and determine their stability via linear stability analysis. **[4]**

(b) Plot the phase portrait of the system in Part (a), indicating the fixed points and their type. **[3]**

(c) Using your answer to Part (b), sketch the graph of the solution $x(t)$ for various initial conditions. **[4]**

(d) Recall that the solution to the Logistic equation $\dot{N} = rN(1 - N/K)$, where r and K are parameters, is given by

$$N(t) = \frac{KN_0 e^{rt}}{K + N_0(e^{rt} - 1)},$$

where $N_0 = N(0)$. Using this, or otherwise, find a solution $x(t)$ to the system in Part (a) in terms of $x_0 = x(0)$. **[5]**

(e) Using the form of $x(t)$ from Part (d), show that for an initial condition $x_0 > 1$ the solution $x(t)$ 'blows-up' in finite time, i.e. $x(t)$ reaches ∞ for some $t < \infty$. **[4]**

(f) Find and sketch the potential for each of the following dynamical systems. Indicate the fixed points on each sketch.

(i) $\dot{x} = x^3 - x$ **[4]**

(ii) $\dot{x} = xe^{-x^2}$. **[4]**

Question 2. [28 marks] (**Bifurcations**) Consider the following dynamical system

$$\dot{x} = xr - x\tan(x), \quad -\frac{\pi}{2} < x < \frac{\pi}{2},$$

which has an r-independent fixed point $x^* = 0$ for all r.

(a) Find any other fixed points in terms of the control parameter r. **[2]**

(b) Find the bifurcation point (x^*, r_c), Taylor expand about this point to get the normal form and identify the type of bifurcation. **[8]**

(c) Using linear stability analysis find the stability of the trivial fixed point $x^* = 0$ for the parameter ranges $r < r_c$ and $r > r_c$. **[4]**

(d) Using the results from Parts (a), (b) and (c), or otherwise, sketch

(i) The phase portraits for $r < r_c$, $r = r_c$ and $r > r_c$. **[6]**

(ii) The corresponding bifurcation diagram. **[4]**

(e) Suppose we add an imperfection parameter h, so that our system becomes

$$\dot{x} = h + xr - x\tan(x), \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Show that for fixed $h > 0$ no bifurcation occurs as r is varied. **[4]**

Question 3. [44 marks] (Two-dimensional systems)

(a) Classify the fixed point $\boldsymbol{x}^* = (0,0)$ for each of the following two-dimensional linear systems and state, with justification, whether $\boldsymbol{x}^*$ is either attracting, Liapunov stable, asymptotically stable, or neither.

(i) $\dot{x} = x + 3y, \dot{y} = 1 + 2y$. **[4]**

(ii) $\dot{x} = 4x + y, \dot{y} = -3x$. **[4]**

(iii) $\dot{x} = -2x, \dot{y} = x - 2y$. **[4]**

(b) Consider the following two-dimensional system

$$\dot{x} = x^3 - x\,, \quad \dot{y} = y + 1 - e^x\,.$$

(i) Identify the fixed points and classify the type of each fixed point by performing a linear stability analysis. **[6]**

(ii) Find equations for all the nullclines; sketch these in the phase plane, indicating the direction of motion along each nullcline. **[6]**

(iii) Using Parts (i) and (ii) sketch the entire phase portrait, indicating typical trajectories and the direction of motion along these trajectories. **[6]**

(c) Consider the following conservative system

$$\ddot{x} = rx - e^x =: -\frac{dV(x)}{dx}, \quad r > 0\,. \tag{1}$$

(i) Find an expression for $V(x)$ and show that for $0 < r < e$ the potential $V(x)$ has no stationary points and for $r > e$ the potential $V(x)$ has two stationary points. **[4]**

(ii) Sketch the graph of $V(x)$ when $0 < r < e$ and $r > e$. **[3]**

(iii) Perform a transformation to turn the second-order equation in (1) into a system of two coupled first-order differential equations and find the associated conserved quantity. **[2]**

(iv) Using Parts (i), (ii) and (iii) sketch the phase portrait when $0 < r < e$ and $r > e$, indicating typical trajectories, the direction of motion along these trajectories and, where appropriate, any homoclinic orbits. **[5]**

End of Paper.