Queen Mary University of London

Main Examination period 2018

MTH742P/U: Advanced Combinatorics

Duration: 3 hours

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Examiners: J. N. Bray and M. Jerrum

In any question, you may use earlier results in later parts. For example, you may use part (c) in part (f), but not part (d) in part (c).

Standard notation is used for graphs, namely: C_n (cycle), E_n (empty), K_n (complete), $\mathrm{K}_{r,s}$ (complete bipartite), P_k (path with k edges).
We use $\mathbb{P}$ for probability and $\mathbb{E}$ for expectation. Recall also that $0 \in \mathbb{N}$.

Question 1. [25 marks] In this question n, r, s, k are non-negative integers such that $r \geqslant 1$, $n = kr + s$ and $0 \leqslant s < r$.

(a) Define what it means for a graph to be ***r*-partite**. And what does it mean for an r-partite graph to be **complete** r-partite? **[4]**

(b) Define the **Turán graphs** $\mathrm{T}_r(n)$. **[2]**

(c) Give alternative notations for the graphs $\mathrm{T}_1(n)$, $\mathrm{T}_2(n)$ and $\mathrm{T}_r(n)$ for $r \geqslant n$, using the symbols E and K. **[3]**

(d) Let $t_r(n) = \mathrm{e}(\mathrm{T}_r(n))$ be the number of edges in $\mathrm{T}_r(n)$. Show that

$$t_r(n) = t_r(kr+s) = k^2\binom{r}{2} + ks(r-1) + \binom{s}{2}.$$

[5]

(e) Hence, or otherwise, prove for $n \geqslant r$ that

$$t_r(n) - t_r(n-r) = \binom{r}{2} + (r-1)(n-r).$$

[3]

Turán's Theorem states (among other things) that a K_{r+1}-free graph on n vertices has at most $t_r(n)$ edges. In one proof of this, we fix r and induct on n, and for $n > r$ we consider a K_{r+1}-free graph Γ maximising the number of edges. Within Γ, we locate an r-clique A, and let $B = \Gamma \setminus A$.

(f) Establish the base cases of Turán's Theorem, namely those for which $n \leqslant r$. **[2]**

(g) Find equalities or upper bounds for $\mathrm{e}(A)$, $\mathrm{e}(B)$ and $\mathrm{e}(A, B)$ and thus get an upper bound for $\mathrm{e}(\Gamma)$. **[6]**

Question 2. [25 marks]

(a) Let $\Gamma = (V,E)$ be a simple graph, and let $f : V \to \mathbb{R}$ be a function. Prove the following generalisation of the Handshaking Lemma:

$$\sum_{\{u,v\}\in E} (f(u)+f(v)) = \sum_{v\in V} f(v)\mathrm{d}(v),$$

where $\mathrm{d}(v)$ denotes the degree of the vertex v. **[5]**

(b) Deduce the following:

$$2|E| = \sum_{v\in V} \mathrm{d}(v) \qquad \text{and} \qquad \sum_{\{u,v\}\in E} (\mathrm{d}(u)+\mathrm{d}(v)) = \sum_{v\in V} \mathrm{d}(v)^2.$$

[3]

(c) State Mantel's Theorem on the maximum number of edges in a triangle-free graph on n vertices. State the isomorphism type(s) of all graphs meeting the bound. **[5]**

(d) For each integer $n \geqslant 3$, prove that the number of edges in an n vertex graph with exactly one triangle is at most

$$\left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 2.$$

[Hint: Let Γ be an n vertex graph with exactly one triangle, let $T = \{u,v,w\}$ be the triangle in Γ, and let $\Delta = \Gamma \setminus T$. Use this notation to help you construct a proof.] **[9]**

(e) For each integer $n \geqslant 3$ construct a graph with exactly one triangle meeting the bound of the previous part. No proofs are required. **[3]**

Question 3. [25 marks]

(a) Define the notion of a **projective plane**. [Use the definition I gave in lectures.] **[4]**

(b) Define what it means for a projective plane to have **order** d. **[2]**

(c) For each prime power $q > 1$ give a construction of a projective plane of order q. Prove that what you make is indeed a projective plane with the required properties. **[8]**

(d) Starting from an arbitrary projective plane of order d, construct a C_4-free graph (from which one can recover the original projective plane), and show the graph is indeed C_4-free. Determine, with justification, the number of vertices and edges of that graph. **[7]**

(e) Let the graph constructed in the previous part have n vertices. Determine its number m of edges in terms of n, and state what happens asymptotically. **[4]**

Question 4. [25 marks]

(a) Define what is meant by a **Hamiltonian cycle** in a finite graph. [You need not define a cycle.] **[2]**

(b) State Dirac's Theorem on Hamiltonian cycles in graphs. **[3]**

(c) For each integer $n \geqslant 3$, give an example of a graph with n vertices and $\binom{n}{2} - n + 2$ edges, which contains no Hamiltonian cycle. Justify your answer. **[3]**

(d) For each integer $n \geqslant 3$, give an example of a **bipartite** graph with n vertices and at least $\frac{1}{4}n^2 - 1$ edges, which contains no Hamiltonian cycle. Justify your answer. **[4]**

(e) Let P_k denote the path with k edges. Let $k, n \in \mathbb{N}^+$ be such that n is a multiple of k. Give an example of a graph Γ with n vertices and $\frac{1}{2}(k-1)n$ edges, which is P_k-free. **[3]**

(f) State a theorem on the maximum possible number of edges of a P_k-free graph with n vertices, which is best possible whenever n is a multiple of k. **[3]**

(g) Prove this theorem (by induction on n, or otherwise). You may assume that if $k, n \in \mathbb{N}^+$ with $k < n$, and if Γ is a connected graph with $n \geqslant 3$ vertices and with minimum degree $\delta(\Gamma) \geqslant k/2$, then Γ contains a path with k edges. **[7]**

Question 5. [25 marks]

(a) Let $n > k > 0$ be integers. Recall that a tournament has property S_k if for any subset of k vertices there is some vertex that defeats them all. Fix $V = \{1, \ldots, n\}$, and consider the set of all $2^{\binom{n}{2}}$ tournaments on V, with each being equally likely.

Prove that if $\binom{n}{k}(1 - 2^{-k})^{n-k} < 1$ then there is a tournament on V with property S_k. [Hint: For suitable $K \subseteq V$ consider the probability of the event A_K that no vertex beats every member of K.] **[10]**

Let $\Gamma = (V, E)$ be a graph with n vertices and m edges. For each $v \in V$ choose it with probability $\frac{1}{2}$, independently of the other vertices, let L be the set of chosen vertices, and let $R = V \setminus L$. For $e = \{u, v\} \in E$ let $X_e = X_{\{u,v\}}$ be the indicator random variable that one endpoint of e lies in L and the other in R.

(b) For $e \in E$ calculate $\mathbb{P}(X_e = 1)$ and $\mathbb{E}X_e$. **[8]**

(c) Prove that Γ contains a bipartite subgraph with at least $\frac{m}{2}$ edges. [Hint: Consider a suitable random variable.] **[7]**

End of Paper.