MTH742U/MTH742P: Advanced Combinatorics

Duration: 3 hours

Date and time: 19 May 2016, 10:00 am

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions answered will be counted.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): David Ellis

Question 1. (a) Define what is meant by a *Hamiltonian cycle* in a finite graph. (You need not define a cycle.) [**2**]

(b) State Dirac's theorem on Hamiltonian cycles in graphs. [**3**]

(c) For each integer $n \geq 3$, give an example of a graph with n vertices and $\binom{n}{2} - n + 2$ edges, which contains no Hamiltonian cycle. Justify your answer. [**3**]

(d) For each integer $n \geq 3$, give an example of a *bipartite* graph with n vertices and at least $\frac{1}{4}n^2 - 1$ edges, which contains no Hamiltonian cycle. Justify your answer. [**5**]

(e) Let P_k denote the path with k edges. Let $k, n \in \mathbb{N}$ be such that n is a multiple of k. Give an example of a graph G with n vertices and $\frac{1}{2}(k-1)n$ edges, which is P_k-free. [**3**]

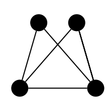
(f) State a theorem on the maximum possible number of edges of a P_k-free graph with n vertices, which is best-possible whenever n is a multiple of k. [**3**]

(g) Prove this theorem (by induction on n, or otherwise). You may assume that if $k, n \in \mathbb{N}$ with $k < n$, and if H is a connected graph with n vertices and with minimum degree $\delta(H) \geq k/2$, then H contains a path with k edges. [**6**]

Question 2. (a) State Mantel's theorem on the maximum possible number of edges of a triangle-free graph with n vertices. For each $n \in \mathbb{N}$, describe the graphs for which equality holds in Mantel's theorem. [**5**]

(b) Let G be a finite graph, and let $x \in V(G)$. Define the *degree* $d(x)$ of x. [**1**]

For the rest of this question, let F denote the graph below.



(c) Let $n \in \mathbb{N}$ with $n \geq 2$. Let G be a graph with n vertices. Show that if G is triangle-free, then G has a vertex of degree at most $\lfloor n/2 \rfloor$. [**3**]

(d) Now suppose G is an F-free graph with n vertices which contains a triangle. Show that if xyz is a triangle in G, then $d(x) + d(y) + d(z) \leq n + 3$. Deduce that if $n \geq 5$, then G has a vertex of degree at most $\lfloor n/2 \rfloor$. [**5**]

(e) Using part (c), part (d) and induction on n (or otherwise), prove that for each integer $n \geq 4$, any F-free graph with n vertices has at most $\lfloor n^2/4 \rfloor$ edges. [**6**]

(f) For each integer n with $n = 4$ or $n \geq 10$, describe (with justification) the graphs with n vertices for which equality holds in part (e), that is, the F-free graphs with n vertices and exactly $\lfloor n^2/4 \rfloor$ edges. (You may assume the results in part (a).) [**5**]

Question 3. (a) Let G be a finite graph. Define the *chromatic number* $\chi(G)$. [**1**]

(b) Describe the *greedy colouring algorithm* for colouring the vertices of a finite graph. [**2**]

(c) Let G be a finite graph with maximum degree Δ. Show that, when applied to any ordering of the vertices of G, the greedy colouring algorithm produces a proper colouring of the vertices of G using at most $\Delta + 1$ colours. [**3**]

(d) Let G be a finite graph. By applying the greedy colouring algorithm to an appropriate ordering of the vertices of G, or otherwise, show that

$$\chi(G) \leq \max\{\delta(H) : \ H \text{ is a subgraph of } G\} + 1,$$

where $\delta(H)$ denotes the minimum degree of H. [**5**]

(e) Using the result in part (d), or otherwise, prove that for any connected, finite graph which is not regular, we have $\chi(G) \leq \Delta(G)$, where $\Delta(G)$ denotes the maximum degree of G. [**4**]

(f) Let G be a finite, connected graph which is not a clique. Define the *connectivity* $\kappa(G)$ of G. (You do not need to define the term connected, but if you use the term k-connected, you should define it.) [**3**]

(g) Using the result in part (e), or otherwise, prove that a finite graph G with $\kappa(G) = 1$ has $\chi(G) \leq \Delta(G)$. [**5**]

(h) Write down two non-isomorphic, connected graphs G with 5 vertices and with $\chi(G) = \Delta(G) + 1$. [**2**]

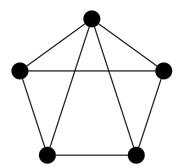
Question 4. Throughout this question, let H be a finite graph with at least one edge.

(a) Define the *Turán number* $\text{ex}(n, H)$ (for each $n \in \mathbb{N}$), and define the *Turán density* $\pi(H)$. [**4**]

(b) State Turán's theorem on the maximum possible number of edges of a K_{r+1}-free graph with n vertices. For each $r, n \in \mathbb{N}$ with $n \geq r \geq 2$, describe the graphs for which equality holds in Turán's theorem. [**6**]

(c) Let $\chi(H)$ denote the chromatic number of the graph H. State a formula for $\pi(H)$ in terms of $\chi(H)$. [**2**]

(d) Using part (c), or otherwise, calculate the Turán density of the graph below. Justify your answer. [**4**]



(e) Show that for each integer $r \geq 2$, there exists a finite graph A_r such that $\chi(A_r) = r + 1$ but $\text{ex}(n, A_r) > \text{ex}(n, K_{r+1})$ for all integers $n > r$. (You may assume any result in parts (a)-(d).) [**4**]

(f) Show that for each integer $r \geq 2$, there exists a finite graph B_r which is K_{r+1}-free but has $\pi(B_r) = 1 - 1/r$. (You may assume any result in parts (a)-(d).) [**5**]

Question 5. (a) Define what is meant by the *Erdős-Rényi random graph* $G_{n,p}$. [**4**]

(b) Show that the number of copies of C_4 in $K_{\{1,2,\ldots,n\}}$ is

$$\tfrac{1}{8}n(n-1)(n-2)(n-3).$$

[**4**]

(c) Let X denote the number of copies of C_4 in $G_{n,p}$. Write down a formula for $\mathbb{E}[X]$ in terms of n and p. [**2**]

For the rest of this question, fix $p = n^{-2/3}$.

(d) Show that

$$\text{Prob}\{X \geq \tfrac{1}{4}n^{1/3}(n-1)\} \leq \tfrac{1}{2}.$$

You may assume Markov's inequality, which says that for any non-negative random variable R with finite mean, and any $a > 0$, we have

$$\text{Prob}\{R \geq a\} \leq \frac{\mathbb{E}[R]}{a}.$$

[**3**]

(e) Let Y denote the number of edges of $G_{n,p}$. Show that

$$\mathbb{E}[Y] = \tfrac{1}{2}n^{1/3}(n-1).$$

(You may assume the standard formula for the mean of a binomial random variable $\text{Bin}(N,p)$.) [**2**]

(f) Recall that for any binomial random variable Z with mean μ, and any $\delta > 0$, we have

$$\text{Prob}\{Z \leq (1-\delta)\mathbb{E}[Z]\} \leq e^{-\delta^2\mu/2}.$$

Use this fact to show that

$$\text{Prob}\{Y \leq \tfrac{3}{8}n^{1/3}(n-1)\} < \tfrac{1}{2}$$

provided n is sufficiently large. [**3**]

(g) Explain how to deduce from parts (d) and (f) that there exists a graph with n vertices, at least $\frac{3}{8}n^{1/3}(n-1)$ edges, and at most $\frac{1}{4}n^{1/3}(n-1)$ copies of C_4, provided n is sufficiently large. [**4**]

(h) Suggest how to modify this graph to produce a C_4-free graph with n vertices and at least $\frac{1}{8}n^{1/3}(n-1)$ edges, for all sufficiently large n. [**3**]

End of Paper.