Queen Mary
University of London

M. Sci. Examination by course unit 2015

MTH734U: Topics in Probability and Stochastic Processes

Duration: 3 hours

Date and time: 12th May 2015, 14:30–17:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Exam papers must not be removed from the examination room.

Examiner(s): Dr Dudley Stark

Question 1. Let $N(t)$, $t \geq 0$, be a continuous time renewal process with interoccurrence times $X_i > 0$ which are independent, identically distributed continuous random variables with common distribution $\mathbb{P}(X_i \leq x) = F(x)$. Let $S_0 = 0$ and let $S_n = X_1 + X_2 + \cdots + X_n$ be the waiting time until the occurrence of the nth event for $n \geq 1$. Suppose $\mu = \mathbb{E}(X_1) < \infty$.

(a) State the theorem about $\lim_{t\to\infty} \frac{N(t)}{t}$ proved in lectures. [**3**]

(b) Prove that for all integers $n \geq 1$ and all real numbers $t > 0$,

$$\mathbb{P}(N(t) = n) = \mathbb{P}(S_n \leq t) - \mathbb{P}(S_{n+1} \leq t).$$

[**5**]

(c) The current life δ_t has limiting distribution

$$\lim_{t\to\infty} \mathbb{P}(\delta_t \leq x) = \frac{1}{\mu}\int_0^x [1 - F(u)]\, du.$$

Suppose that a renewal process has interoccurrence distribution $X_i \sim \text{Uniform}(0, 1)$.

(i) Find $\lim_{t\to\infty} \mathbb{P}(\delta_t \leq x)$ for all $x \geq 0$. [**8**]

(ii) Let $N_{\text{D}}(t)$ be the stationary renewal process corresponding to $N(t)$. Find $M_{\text{D}}(t) = \mathbb{E}(N_{\text{D}}(t))$, stating any theorems from lectures that you use. [**4**]

Question 2.

(a) Given a semi-Markov process on states $\{1, 2, \ldots, N\}$, suppose that when the process enters state i, it stays there a random amount of time having expectation μ_i after which it jumps to state j with probability $P_{i,j}$.

(i) State what is meant by the discrete time Markov chain X_n, where n can take values $n = 0, 1, 2, \ldots$, associated with the semi-Markov process. [**2**]

(ii) Suppose that $\pi_i = \lim_{n\to\infty} \mathbb{P}(X_n = i)$ exists, where X_n is the discrete time Markov chain from part (i). Show that for all $0 \leq j \leq N$,

$$\pi_i = \sum_{j=0}^{N} \pi_j P_{j,i}.$$

[**5**]

(b) Suppose a machine can be in one of three states: Good, Fair or Bad. After entering state Good, the machine remains there for a random time having expectation of one year, after which it enters state Fair; after entering state Fair, the machine remains there for a random time having expectation of six months, after which it enters state Bad; after entering state Bad, the machine remains there a random time having expectation of six months, after which it either enters state Good with probability one half or returns to state Bad with probability one half.

(i) In the long run, what is the proportion of transitions to states Good, Fair and Bad, respectively? [**7**]

(ii) In the long run, what is the proportion of time the machine is in the states Good, Fair and Bad, respectively? [**6**]

Question 3.

(a) Ms. Jones is waiting to make a phone call at a train station. There are two public telephone booths next to each other, occupied by Ms. Smith and Ms. West. The duration of each phone call is an Exponential(θ) distributed random variable with $\theta = 1/8$ minutes^{-1}, and durations of phone calls are independent of each other.

(i) Find the probability that among Ms. Jones, Ms. Smith, and Ms. West, Ms. Jones will be the last to finish her call. [**4**]

(ii) Find the expected time from when Ms. Jones arrives to when Ms. Jones, Ms. Smith, and Ms. West have all completed their calls. [**7**]

(b) Let S_i denote the time of the ith arrival of a Poisson process $N(t)$, $t \geq 0$, with rate $\theta > 0$. By conditioning on the value of $N(t)$, find

$$\mathbb{E}\left(\sum_{i=1}^{N(t)} S_i^3\right).$$

as a function of θ and t. [**9**]

Question 4.

(a) Let $\mathbf{G}$ be the generator of a continuous time Markov chain $X(t)$, with $t \geq 0$, and let $\mathbf{P}(t)$ be the matrix such that $\mathbf{P}(t)_{i,j} = \mathbb{P}(X(s+t) = j | X(s) = i)$.

(i) State the equation for $\mathbf{P}(t)$ in terms of $\mathbf{G}$. [**2**]

(ii) State the backwards and forwards Kolmogorov equations. [**2**]

(b) Consider a fleet of three buses. Each bus breaks down independently at rate μ, after which it is sent to the depot for repairs. The repair shop can repair up to two buses at a time and each bus takes an exponential amount of time with parameter λ to repair.

(i) Find the generator for $X(t)$, the number of buses in service. [**8**]

(ii) Find the limiting probability

$$\pi_0 = \lim_{t\to\infty} \mathbb{P}(X(t) = 0).$$

[**8**]

Question 5. Let $B(t)$ be standard Brownian motion with $B(0) = 0$.

(a) State what is meant by the independent increments property. [**3**]

(b) Let $t > s > 0$ be real numbers. Determine the distribution of $B(s) + 2B(t)$. [**6**]

(c) Determine whether the process defined by $\tilde{B}(t) = \sqrt{\alpha t}B(1/\alpha)$, where α is a positive constant, is standard Brownian motion. [**4**]

(d) Let $\tau_x = \min\{u \geq 0 : B(u) = x\}$ and let $M(t) = \max_{0\leq u\leq t} B(u)$. For $x > 0$, use the fact that

$$\mathbb{P}(M(t) \geq x) = 2\mathbb{P}(B(t) > x)$$

to show that the probability density function of τ_x is

$$f_{\tau_x}(t) = \frac{xt^{-3/2}}{\sqrt{2\pi}} e^{-x^2/(2t)} \quad \text{for } 0 < t < \infty.$$

[**7**]

End of Paper.